EUCLID

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THE ELEMENTS OF EUCLID

FOR THE USE OF SCHOOLS AND COLLEGES

WITH NOTES, AN APPENDIX, AND EXERCISES BY

I. TODHUNTER, D.Sc., F.R.S.

NEW EDITION, REVISED AND ENLARGED, BY

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BOOKS I.—IV.

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PREFACE TO THE COMPLETE EDITION.

In preparing a new edition of the late Dr. Todhunter's *Euclid*, the following are the principal alterations and additions that I have made:

(1) The text of the Propositions has been simplified and shortened. A sparing use has been made of symbols in the place of constantly recurring words. Some of the proofs (*e.g.* II. 13) have been altered in accordance with modern usage, but this has always been done subject to the regulations at present in force in the Universities of Oxford and Cambridge.

(2) Considerable trouble has been taken to arrange the book so that, with very few exceptions, each Proposition is commenced on a fresh page, and may be read by the student without his turning over a leaf.

(3) The proofs of Book V. have been much shortened. To them, and also to the proofs of Book II., the corresponding algebraic formulae have been added.

(4) The more important of Dr. Todhunter's Notes have been appended to the Propositions to which they refer.

(5) The total number of Exercises has been doubled.

(6) The easier of the Exercises in the previous edition, and a large number of additional ones, have been classified and follow, in the Text, the Propositions on which they depend. The more important of these have asterisks prefixed to them, and, as far as possible, their results should

be remembered by the student as part of his geometrical knowledge.

(7) All through the book, with the exception of the Exercises at the end, which are left to the student, hints have been annexed to the more difficult and more important Exercises.

(8) The Appendix has been more than doubled in quantity, and the theorems in it have been classified according to the Book to which they refer and on which they depend; a large number of Exercises has been incorporated in it.

(9) Sections have been added dealing with Poles and Polars, Orthogonal Circles, Pedal Triangles, The Pedal Line, The Nine-Point Circle, Co-axal Circles, Harmonic Ranges, Inversion, and the Properties of a Complete Quadrilateral.

Out of the large number of Propositions on these subjects it is clear that only a selection could be made, but I have endeavoured within the range chosen to omit no important Proposition. Without unduly increasing the size of the book it was impossible to prove all such important Propositions in the text. It is hoped, however, that the hints attached to those given as Exercises will be sufficient for any fairly intelligent student.

For any corrections of misprints or errors, or any suggestions for improvement, I shall be very grateful.

S. L. LONEY.

ROYAL HOLLOWAY COLLEGE, EGHAM,
SURREY, *April 21st*, 1899

CONTENTS.

EUCLID'S ELEMENTS.

BOOK I.

DEFINITIONS.

1. A **point** is that which has position but no magnitude.

2. A **line** is that which has length without breadth.

3. The extremities of a line are points, and the intersection of two lines is a point.

4. A **straight line** is one which lies evenly between its extreme points.

5. A superficies, or **surface**, is that which has only length and breadth.

6. The boundaries of a surface are lines.

7. A plane surface, or a **plane**, is that in which any two points being taken, the straight line between them lies wholly in that surface. ?

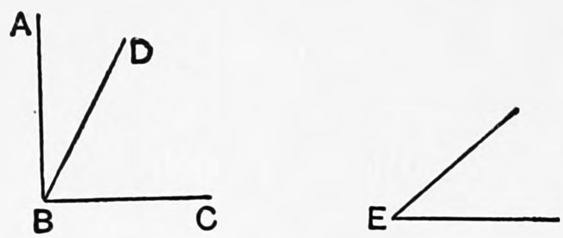
[Thus if we take a piece of wood, one of whose edges is straight, and apply it to the surface, and find that the edge fits closely to the surface everywhere, we know that the surface is a plane.]

8. A **plane angle** is the inclination of two lines to one another in a plane, which meet together, but are not in the same direction.

9. A **plane rectilineal angle** is the inclination of two straight lines to one another, which meet together, but are not in the same straight line. ?

The point where the two straight lines meet is called the **vertex** of the angle, and the straight lines themselves are sometimes called the **arms**.

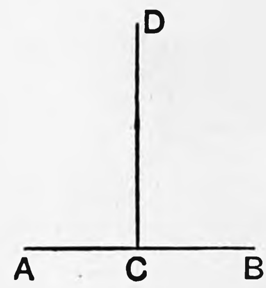
Note. When several angles are at one point B, any one of them is expressed by three letters, of which the letter which is at the vertex of the angle is put between the other two letters, and one of these two letters is somewhere on one of those straight lines, and the other letter on the other straight line. Thus, the angle which is contained



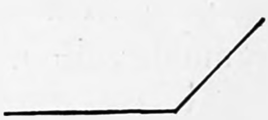
by the straight lines AB, CB is named the angle ABC, or CBA; the angle which is contained by the straight lines AB, DB is named the angle ABD, or DBA; and the angle which is contained by the straight lines DB, CB is named the angle DBC, or CBD; but if there be only one angle at a point, it may be expressed by a letter placed at that point, as the angle E.

Two such angles as ABD, DBC, which are on opposite sides of one common bounding line BD, are called **adjacent** angles.

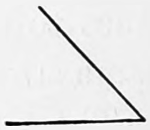
The beginner must carefully observe that no change is made in an angle by prolonging the lines that form it, that is, by altering the length of its arms.



10. When a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is called a **right angle**; and the straight line which stands on the other is called a **perpendicular** to it.



11. An **obtuse angle** is an angle which is greater than a right angle.

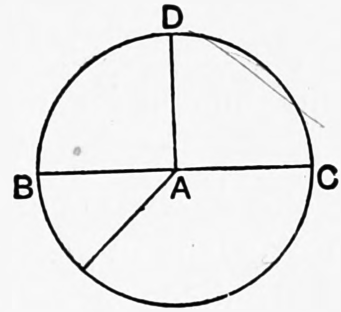


12. An **acute angle** is an angle which is less than a right angle.

13. A **plane figure** is one which is enclosed by one or more bounding lines, straight or curved; and the sum of these bounding lines is called a **perimeter**.

14. If the boundaries consist of straight lines only, it is called a **plane rectilineal figure**, and these straight lines are called its **sides.**

15. A **circle** is a plane figure contained by one line, which is called the **circumference**, and is such, that all straight lines drawn from a certain point within the figure to the circumference are equal to one another:



Also this point is called the **centre** of the circle.

16. A **radius** of a circle is a straight line drawn from the centre to the circumference.

17. A **diameter** of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

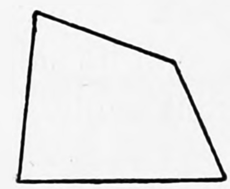
18. A **semicircle** is the figure contained by a diameter and the part of the circumference cut off by the diameter.

[Thus, in the above figure, A is the centre; AB, AD, AC are radii; BC is a diameter; and the figure bounded by the straight line BC and the curved line BDC is a semicircle.]

19. A **segment** of a circle is the figure contained by a straight line and the part of the circumference which it cuts off.

20. A **triangle** is a plane figure contained by three straight lines. [Any angular point may be called a **vertex** and the opposite side the **base.**]

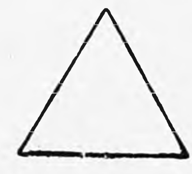
21. A **quadrilateral** is a plane figure contained by four straight lines.



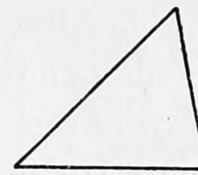
[The straight line joining two opposite corners of a quadrilateral is called a **diagonal.**]

22. A **polygon** is a plane figure contained by more than four straight lines.

23. An **equilateral triangle** is one which has three equal sides.



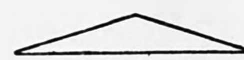
24. An **isosceles triangle** is one which has two sides equal.



25. A **scalene triangle** is one which has three unequal sides.

26. A **right-angled triangle** is one which has a right angle.

[The side opposite to the right angle in a right-angled triangle is frequently called the **hypotenuse.**]

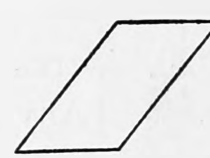


27. An **obtuse-angled triangle** is one which has an obtuse angle.

28. An **acute-angled triangle** is one which has three acute angles.

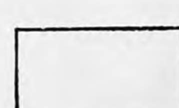
[It will be seen later that every triangle has at least two acute angles.]

29. **Parallel straight lines** are such as are in the same plane, and which being produced ever so far both ways do not meet.



30. A **parallelogram** is a four-sided figure which has its opposite sides parallel.

31. A **square** is a four-sided figure which has all its sides equal, and one of its angles a right angle.



32. A **rectangle** is a parallelogram which has one of its angles a right angle.

[It will be shewn (see Note, Prop. 46) that all the angles of a rectangle or a square are right angles.]

33. A **rhombus** is a four-sided figure which has all its sides equal, but its angles are not right angles.

34. A rhomboid is a four-sided figure which has its opposite sides equal to one another, but all its sides are not equal, nor its angles right angles.

35. A trapezium is a four-sided figure which has two of its sides parallel.

POSTULATES.

Let it be granted,

1. That a straight line may be drawn from any one point to any other point:

2. That a terminated straight line may be produced to any length in a straight line: and

3. That a circle may be described with any centre, at any distance from that centre, that is, with any given line drawn from the centre as radius.

Note. The postulates state what processes we assume that we can effect. It is sometimes stated that the postulates amount to requiring the use of a *ruler* and *compasses.* It must, however, be observed that the ruler is not supposed to be a *graduated* ruler, so that we cannot use it to measure off assigned lengths. Also, we are not supposed to use the compasses for the purpose of transferring any distance from one part of a figure to another; in other words, the compasses may be supposed to close of themselves, as soon as one of their points is removed from the paper. [After Prop. 3 it will be found that this restriction is no longer necessary.]

AXIOMS.

1. Things which are equal to the same thing are equal to one another.

2. If equals be added to equals the wholes are equal.

3. If equals be taken from equals the remainders are equal.

4. If equals be added to unequals the wholes are unequal.

5. If equals be taken from unequals the remainders are unequal.

6. Things which are double of the same thing are equal to one another.

7. Things which are halves of the same thing are equal to one another.

8. The whole is greater than its part.

9. Magnitudes which can be made to coincide with one another are equal to one another.

[This method of placing one geometrical magnitude upon a second is called the method of **superposition**, and the first magnitude is said to be **applied** to the other.]

10. Two straight lines cannot enclose a space.

This axiom should be extended thus:
If two straight lines coincide in two points, they must coincide both beyond and between these points.

11. All right angles are equal to one another.

[This axiom admits of proof; see Note to I. 14.]

12. If a straight line meet two straight lines, so as to make the two interior angles on the same side of it together less than two right angles, these straight lines, being continually produced, shall at length meet on that side on which are the angles which are less than two right angles.

Note. An axiom is a truth which can be taken for granted, and requires no proof. The axioms are called in the original *Common Notions.*

The first eight are true of magnitudes of all kinds, and are sometimes called General Axioms; the remainder refer exclusively to geometrical magnitudes, and are sometimes called Geometrical Axioms.

The fourth axiom is sometimes referred to in editions of Euclid when in reality more is required than this axiom expresses. Euclid says that if A and B be unequal, and C and D equal, the sum of A and C is *unequal* to the sum of B and D. What Euclid often requires is something more, namely, that if A be greater than B, and C and D be equal, the sum of A and C is *greater* than the sum of B and D. Such an axiom as this is required, for example, in I. 17. A similar remark applies to the fifth axiom.

The eleventh axiom is not, strictly speaking, an axiom, for it can be proved; it is not required before I. 14, and the twelfth axiom is not required before I. 29; we shall not consider these axioms until we arrive at the propositions in which they are respectively required for the first time.

NOTE ON EUCLID'S PROPOSITIONS.

Euclid divides his different books into separate propositions, each proposition being derived from previous propositions. Propositions are of two kinds, **Problems** and **Theorems.**

In a problem Euclid states some definite construction which is to be made, such as to draw some particular figure. When this construction has been made the problem is *solved.*

In a theorem Euclid states some definite geometrical fact which he proceeds to prove. He names first the **Hypothesis**, or the conditions which he assumes, and then the **Conclusion** which he asserts will follow.

In a proposition we usually have:

(1) The *General Enunciation,* which is a preliminary statement.

(2) The *Particular Enunciation,* which applies the general enunciation to a particular diagram.

(3) The *Construction,* which shows what drawing of lines, etc., he wants.

(4) The *Demonstration,* or *Proof,* which shows that the problem has been solved, or that the theorem is true.

The letters Q.E.F. at the end of a problem stand for *Quod erat Faciendum,* that is, *which was to be done.*

The letters Q.E.D. at the end of a theorem stand for *Quod erat Demonstrandum*, that is, *which was to be proved.*

A **Corollary** is a statement which follows immediately from the proposition to which it is appended, and which thus requires no further proof.

Many of the corollaries are not in the original text but have been introduced by various editors.

SYMBOLS AND ABBREVIATIONS.

The following symbols are often used for words and expressions that continually occur in Euclid's Proofs. They may be used by the student in writing out propositions, and some of the more common will be gradually introduced in this Edition.

$\therefore$ *for* Therefore.

$=$ *for* Equals *or* Is equal to *or* Are equal to.

$>$ *for* Is greater than. $<$ *for* Is less than.

$\angle$ and $\angle^{s.}$ *for* Angle *and* Angles.

$\triangle$ and $\triangle^{s.}$ *for* Triangle *and* Triangles.

$\perp^{r.}$ *for* Perpendicular to.

$\|^{l.}$ *for* Parallel. $\|^{m.}$ and Par$^{m.}$ *for* Parallelogram.

Rt. $\angle$ *for* Right Angle. $\odot$ *for* Circle *or* Circumference.

The symbol $\because$ is sometimes used for Because. It is, however, liable to be confounded with the symbol for Therefore, and will not be used in this Edition.

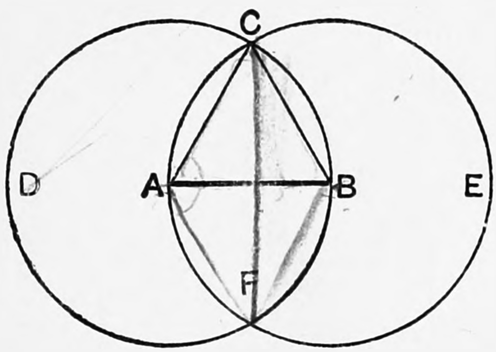
In addition any well understood abbreviations for words may be used; *e.g.*:

Def. *for* Definition; Ax. *for* Axiom; Post. *for* Postulate; Constr. *for* Construction; Hyp. *for* Hypothesis; Pt. *for* Point; Str. *for* Straight; Perp. *for* Perpendicular; Alt. *for* Altitude; Sq. *for* Square; Rect. *for* Rectangle; Quad$^{l.}$ *for* Quadrilateral; Adj. *for* Adjacent.

PROPOSITION 1. PROBLEM.

To describe an equilateral triangle on a given finite straight line.

Let AB be the given straight line:
it is required to describe an equilateral triangle on AB.



Construction. With centre A and radius AB, describe the circle BCD. [*Postulate* 3.

With centre B and radius BA, describe the circle ACE. [*Postulate* 3.

From the point C, at which the circles cut one another, draw the straight lines CA and CB. [*Postulate* 1.

ABC shall be an equilateral triangle.

Proof. Because A is the centre of the circle BCD,

AC is equal to AB. [*Definition* 15.

Also because B is the centre of the circle ACE,

BC is equal to BA. [*Definition* 15.

But it has been shewn that CA is equal to AB;
therefore CA and CB are each of them equal to AB.
But things which are equal to the same thing are equal to one another; [*Axiom* 1.

therefore CA is equal to CB.

Therefore CA, AB, BC are equal to one another.

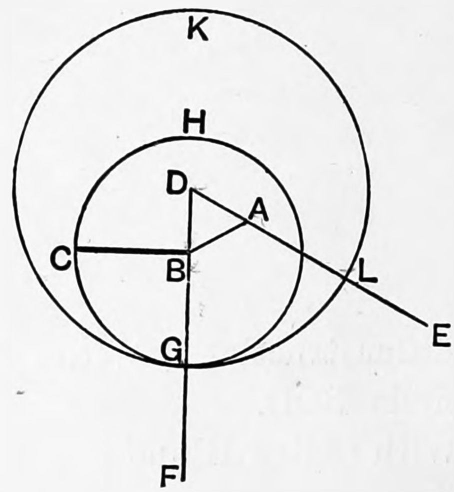
Wherefore *the triangle* ABC *is equilateral,* [*Definition* 23.
and it is described on the given straight line AB. [Q. E. F.

PROPOSITION 2. PROBLEM.

From a given point to draw a straight line equal to a given straight line.

Let A be the given point, and BC the given straight line: *it is required to draw from the point* A *a straight line equal to* BC.

Construction. Join AB, [*Post.* 1.
and on it describe the equilateral triangle DAB, [I. 1.
and produce the straight lines DA, DB to E and F. [*Postulate* 2.
With centre B and radius BC, describe the circle CGH,
meeting DF at G. [*Postulate* 3.
With centre D and radius DG, describe the circle GLK,
meeting DE at L. [*Postulate* 3.
AL shall be equal to BC.



Proof. Because the point B is the centre of the circle CGH,
BC is equal to BG. [*Definition* 15.
Also because D is the centre of the circle GLK,
DL is equal to DG; [*Definition* 15.
and DA, DB parts of them are equal; [*Definition* 23.
therefore the remainders AL, BG are equal. [*Axiom* 3.
But it has been shewn that BC is equal to BG;
therefore AL and BC are each of them equal to BG.
But things which are equal to the same thing are equal to one another. [*Axiom* 1.

Therefore AL is equal to BC.

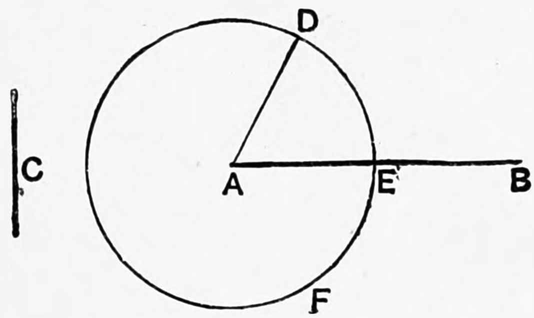
Wherefore *from the given point* A *a straight line* AL *has been drawn equal to the given straight line* BC. [Q.E.F.

PROPOSITION 3. PROBLEM.

From the greater of two given straight lines to cut off a part equal to the less.

Let AB and C be the two given straight lines, of which AB is the greater:

It is required to cut off from AB, *the greater, a part equal to* **C**, *the less.*



Construction. From the point A draw the straight line AD equal to C; [I. 2.

and with centre A and radius AD, describe the circle DEF, meeting AB at E. [*Postulate* 3.

Then AE shall be equal to C.

Proof. Because A is the centre of the circle DEF,

therefore AE is equal to AD. [*Def.* 15.

But C is equal to AD. [*Construction.*

Therefore AE and C are each of them equal to AD.

Therefore AE is equal to C. [*Axiom* 1.

Wherefore *from* AB, *the greater of two given straight lines, a part* AE *has been cut off equal to* C, *the less.*

EXERCISES.

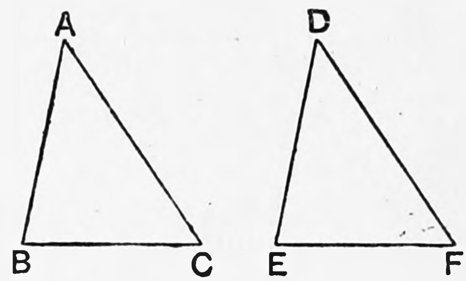
1. In the figure of I. 1, if the two circles meet again in F, prove that ACBF is a rhombus.

2. On a given straight line describe an isosceles triangle, having each of the sides equal to a given straight line.

3. On a given straight line describe an isosceles triangle, having each of the sides double the base.

PROPOSITION 4. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal to one another;
they shall also have their bases or third sides equal;
the two triangles shall be equal in area,
and their other angles shall be equal, each to each, namely, those to which the equal sides are opposite;
that is, the triangles shall be equal in all respects.



Let ABC, DEF be two triangles which have
the side AB equal to the side DE,
the side AC equal to the side DF,
and the angle BAC equal to the angle EDF;
then shall the base BC *be equal to the base* EF,
the triangle ABC *shall be equal in area to the triangle* DEF,
and the other angles shall be equal, each to each, to which the equal sides are opposite, namely, the angle ABC *to the angle* DEF, *and the angle* ACB *to the angle* DFE.

Proof. For if the triangle ABC be applied to the triangle DEF, so that the point A may be on the point D, and the straight line AB on the straight line DE,
the point B will coincide with the point E,
because AB is equal to DE. [*Hypothesis.*
And because AB coincides with DE,
and the angle BAC is equal to the angle EDF, [*Hypothesis.*
therefore AC will fall on DF.

Therefore also the point C will coincide with the point F,
because AC is equal to DF. [*Hypothesis.*

But the point B was shewn to coincide with the point E, therefore the base BC will coincide with the base EF; for if not, two straight lines will enclose a space, which is impossible. [*Axiom* 10.

Therefore the base BC coincides with the base EF, and is equal to it. [*Axiom* 9.

Therefore the whole triangle ABC coincides with the whole triangle DEF, and is equal to it. [*Axiom* 9.

And the other angles of the one coincide with the other angles of the other, and are equal to them, namely,
the angle ABC to the angle DEF,
and the angle ACB to the angle DFE.

Wherefore, *if two triangles, etc.* [Q.E.D.

Note. Triangles, such as ABC and DEF, which are equal in all respects, are said to be **Congruent.**

EXERCISES.

1. If two straight lines, AB and CD, bisect one another at right angles in O, any point P in either of them, AB, is equidistant from the ends, C and D, of the other.

****2.** If the straight line drawn from the vertex of a triangle to the middle point of the base cuts the base at right angles, the triangle is isosceles.

3. D and E are the middle points of the equal sides AB, AC of an isosceles triangle ABC; prove that CD and BE are equal.

4. ABCD is a quadrilateral, and its diagonals bisect one another at right angles; prove that ABCD is a rhombus.

5. The sides AB, AD of a quadrilateral ABCD are equal, and the diagonal AC bisects the angle BAD; prove that the sides CB, CD are equal, and that the diagonal AC bisects the angle BCD.

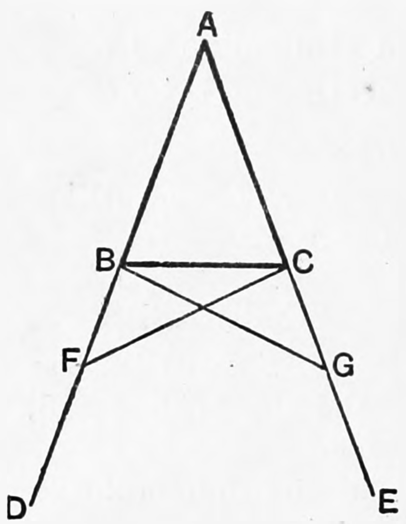
6. ABCD and EFGH are quadrilaterals, such that the sides AB, BC, CD are equal respectively to the sides EF, FG, GH, and the angles ABC, BCD are equal respectively to the angles EFG, FGH. Prove that the quadrilaterals are equal in all respects.

****7.** The straight line which bisects the vertical angle of an isosceles triangle bisects the base, and is also perpendicular to it.

PROPOSITION 5. THEOREM.

The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced, the angles on the other side of the base shall also be equal to one another.

Let ABC be an isosceles triangle, having the side AB equal to the side AC, and let the straight lines AB, AC be produced to D and E:
then the angle ABC *shall be equal to the angle* ACB,
and the angle CBD *to the angle* BCE.



Construction. In BD take any point F,
and from AE the greater cut off AG equal to AF the less. [I. 3.
Join FC and GB.

Proof. (1) In the two triangles AFC, AGB,

because { AF is equal to AG, [*Construction.*
and AC is equal to AB, [*Hypothesis.*
and the contained angle FAG is common to both;

therefore the triangles are equal in all respects, so that
the base FC is equal to the base GB,
the angle ACF is equal to the angle ABG,
and the angle AFC to the angle AGB. [I. 4.

(2) Because the whole AF is equal to the whole AG, of which the parts AB, AC are equal, [*Hypothesis.*
the remainder BF is equal to the remainder CG. [*Axiom* 3.

(3) Then in the triangles BFC, CGB,

because { BF is equal to CG, [*Proved in* (2).
and FC is equal to GB, [*Proved in* (1).
and the angle BFC is equal to the angle CGB; [*Proved in* (1).

therefore the triangles are equal in all respects, [I. 4.
so that the angle FBC is equal to the angle GCB,
and the angle BCF to the angle CBG.

(4) Since the whole angle ABG is equal to the whole angle ACF, [*Proved in* (1).
and that the parts of these, the angles CBG, BCF are also equal; [*Proved in* (3).
therefore the remaining angle ABC is equal to the remaining angle ACB, [*Axiom* 3.
and these are the angles at the base of the triangle ABC.

Also it has been shewn that the angle FBC is equal to the angle GCB, and these are the angles on the other side of the base.

Wherefore, *the angles, etc.* [Q.E.D.

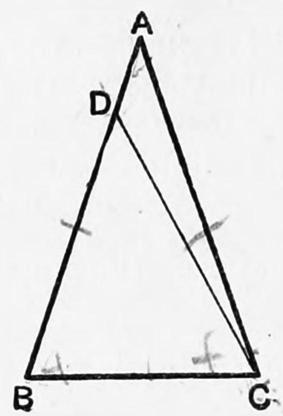
Corollary. Every equilateral triangle is also equiangular.

Note. This proposition is often found hard by beginners. This probably arises from the fact that the triangles ACF, ABG overlap one another; as also do the triangles BCF and CBG. For the part (1) of the proof the student is recommended to draw separately the two triangles ACF and ABG, and consider the figures thus obtained; for the part (3) of the proof he should similarly draw separately the two triangles BCF and CBG.

PROPOSITION 6. THEOREM.

If two angles of a triangle be equal to one another, the sides also which subtend, or are opposite to, the equal angles, shall be equal to one another.

Let ABC be a triangle, having the angle ABC equal to the angle ACB :
then the side AC *shall be equal to the side* AB.



Construction. For if AC be not equal to AB, one of them must be greater than the other.

Let AB be the greater, and from it cut off DB equal to AC the less, [I. 3.

and join DC.

Proof. In the triangles DBC, ACB,

because { DB is equal to AC, [*Construction.*
and BC is common to both triangles,
and the angle DBC is equal to the angle ACB, [*Hypothesis.*

therefore the triangles are equal in all respects, [I. 4.
the less to the greater; which is absurd. [*Axiom* 8.
Therefore AB is not unequal to AC, that is, it is equal to it.

Wherefore, *if two angles, etc.* [Q.E.D.

Corollary. Every triangle, which has three equal angles, is also equilateral.

NOTE TO PROPOSITION 6.

Converse Theorem. One proposition is said to be the converse of another when the conclusion of each is the hypothesis of the other. I. 6 is the *converse* of part of I. 5. Thus in I. 5 the hypothesis is the equality of the sides, and one conclusion is the equality of the angles; in I. 6 the hypothesis is the equality of the angles and the conclusion is the equality of the sides.

The converse of a true proposition is not necessarily true; the student however will see, as he proceeds, that Euclid shews that the converses of many geometrical propositions are true.

I. 6 is an example of the **indirect** mode of demonstration, in which a result is established by shewing that some absurdity follows from supposing the required result to be untrue. Hence this mode of demonstration is called the **reductio ad absurdum.** Indirect demonstrations are often less esteemed than direct demonstrations; they are said to shew that a theorem *is* true rather than to shew *why* it is true. Euclid uses the *reductio ad absurdum* chiefly when he is demonstrating the converse of some former theorem; see I. 14, 19, 25, 40.

EXERCISES.

1. If the angles ABC, ACB at the base of an isosceles triangle ABC are bisected by the straight lines BD and CD, which meet at D, prove that BDC will be an isosceles triangle.

2. BAC is a triangle having the angle B double of the angle A. If BD bisects the angle B and meets AC at D, prove that BD and AD are equal.

In the figure of I. 5, if FC and BG meet at H, prove that

3. BH and CH are equal.

4. AH bisects each of the angles BAC, BHC.

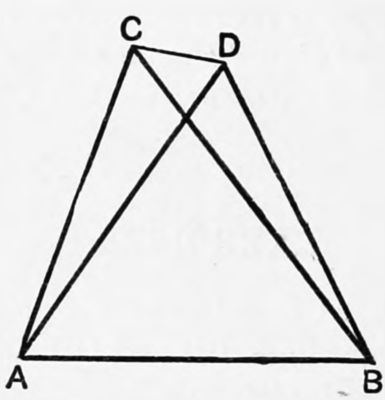
5. AH bisects BC at right angles.

6. If on the sides AB, BC, CA of an equilateral triangle ABC equal lengths AP, BQ, CR be measured, prove that PQR is also an equilateral triangle.

Proposition 7. Theorem.

On the same base, and on the same side of it, there cannot be two triangles having their sides which are terminated at one extremity of the base equal to one another, and likewise those which are terminated at the other extremity equal to one another.

If it be possible, on the same base AB, and on the same side of it, let there be two triangles ACB, ADB, having their sides CA, DA, which are terminated at the extremity A of the base, equal to one another,
and also their sides CB, DB, which are terminated at B, equal to one another.



Construction. Join CD.

Proof. Case I. Let the vertex of each triangle be without the other triangle.

Because AC is equal to AD, [*Hypothesis.*

the angle ACD is equal to the angle ADC. [I. 5.

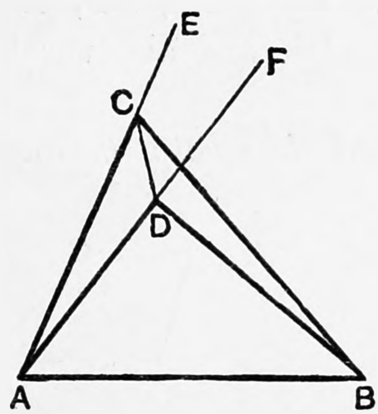
But the angle ACD is greater than the angle BCD, [*Ax.* 8.

therefore the angle ADC is also greater than the angle BCD;

much more then is the angle BDC greater than the angle BCD.

Again, because BC is equal to BD, [*Hypothesis.*
the angle BDC is equal to the angle BCD. [I. 5.
But it has been shewn to be greater, which is impossible.

CASE II. Let one of the vertices as D be within the other triangle ACB, and produce AC, AD to E, F.



Then because AC is equal to AD, in the triangle ACD, [*Hyp.*
the angles ECD, FDC, on the other side of the base CD, are equal to one another. [I. 5.
But the angle ECD is greater than the angle BCD, [*Axiom* 8.
therefore the angle FDC is also greater than the angle BCD; much more then is the angle BDC greater than the angle BCD.
Again, because BC is equal to BD, [*Hypothesis.*
the angle BDC is equal to the angle BCD. [I. 5.
But it has been shewn to be greater, which is impossible.

CASE III. The case in which the vertex D of one triangle is on a side, BC, of the other needs no demonstration.

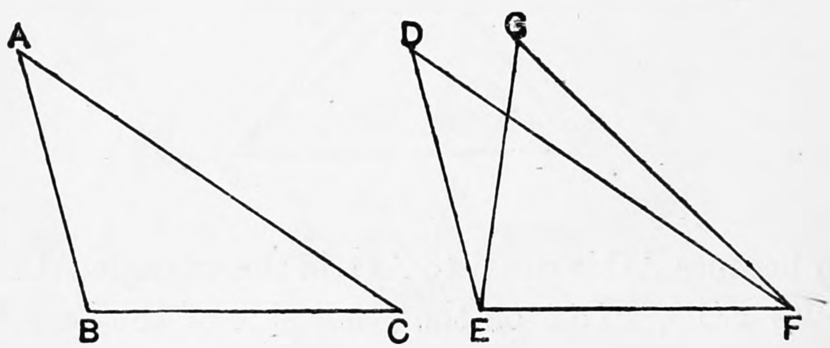
Wherefore, *on the same base, etc.* [Q. E. D.

PROPOSITION 8. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the angle which is contained by the two sides of the one shall be equal to the angle which is contained by the two sides, equal to them, of the other.

Let ABC, DEF be two triangles, having the two sides AB, AC equal to the two sides DE, DF, each to each, namely, AB to DE, and AC to DF, and also the base BC equal to the base EF:

then shall the angle BAC *be equal to the angle* EDF.



Proof. For if the triangle ABC be applied to the triangle DEF, so that the point B may be on the point E,
and the straight line BC on the straight line EF,
the point C will also coincide with the point F,
because BC is equal to EF. [*Hypothesis.*

Therefore, BC coinciding with EF, BA and AC will coincide with ED and DF.

For if not, they must have a different situation as EG, GF; then on the same base and on the same side of it there would be two triangles having their sides DE, GE equal, and also their sides DF, GF equal.

But this is impossible. [I 7.

Therefore the sides BA, AC must coincide with the sides ED, DF.

Therefore also the angle BAC coincides with the angle EDF, and is equal to it. [*Axiom* 9.

Wherefore, *if two triangles, etc.* [Q.E.D.

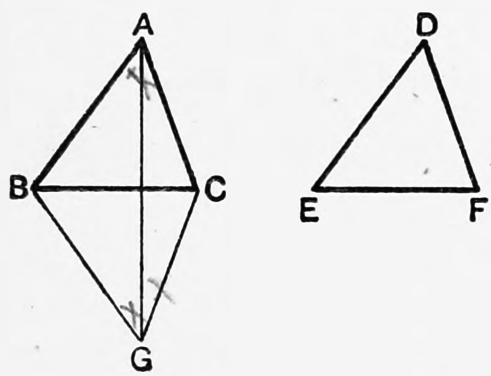
Cor. Since the triangle ABC would coincide with the triangle DEF, these two triangles *are equal in all respects.*

Alternative Proof of Proposition 8.

The following proof is independent of I. 7. It has been recommended by many writers, and is often known as Philo's proof.

Let ABC, DEF be two triangles, having the sides AB, AC equal to the sides DE, DF, each to each, and the base BC equal to the base EF :

the angle BAC *shall be equal to the angle* EDF.



For, let the triangle DEF be applied to the triangle ABC, so that the bases may coincide, the equal sides be conterminous, and the vertices fall on opposite sides of the base.

Let GBC represent the triangle DEF thus applied, so that G corresponds to D. Join AG.

Since, by hypothesis, BA is equal to BG, the angle BAG is equal to the angle BGA. [I. 5.

In the same manner the angle CAG is equal to the angle CGA.

Therefore the whole angle BAC is equal to the whole angle BGC, that is, to the angle EDF.

There are two other cases; for the straight line AG may pass through B or C, or it may fall outside BC: these cases may be treated in the same manner as that which we have considered.

EXERCISES.

**1. If D be the middle point of the base BC of an isosceles triangle ABC, prove that AD is perpendicular to BC.

**2. The opposite angles of a rhombus are equal.

**3. A diagonal of a rhombus bisects each of the angles through which it passes.

**4. The diagonals of a rhombus bisect one another at right angles.

5. ACB and ADB are two triangles on the same side of AB, such that AC is equal to BD, and AD is equal to BC; if AD and BC meet in O, prove that the triangle AOB is isosceles.

ABC is an isosceles triangle and D, E are points in the equal sides AB, AC, such that AD and AE are equal; if BE and CD meet in F, prove that

6. BCF is an isosceles triangle.

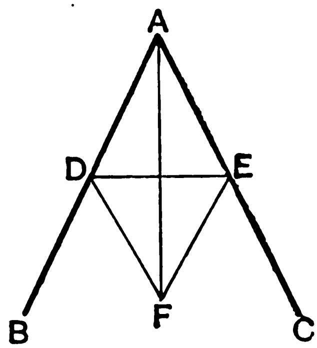
7. AF bisects the angle BAC.

8. AF produced bisects the base BC.

PROPOSITION 9. PROBLEM.

To bisect a given rectilineal angle, that is, to divide it into two equal angles.

Let BAC be the given rectilineal angle:
it is required to bisect it.



Construction. Take any point D in AB, and from AC cut off AE equal to AD; [I. 3.
join DE, and on DE, on the side remote from A, describe the equilateral triangle DEF. [I. 1.
Join AF.
AF shall bisect the angle BAC.

Proof. In the triangles DAF, EAF,

because $\begin{cases} \text{AD is equal to AE,} & \text{[}\textit{Construction.} \\ \text{and AF is common to both,} & \\ \text{and the base DF is equal to the base EF,} & \text{[}\textit{Def.}\text{ 23.} \end{cases}$

therefore the angle DAF is equal to the angle EAF. [I. 8.

Wherefore *the given rectilineal angle* BAC *is bisected by the straight line* AF. [Q.E.F.

Note. The equilateral triangle DEF is to be described on the side *remote* from A; because if it were described on the *same* side, its vertex F might coincide with A and then the construction would fail.

EXERCISES.

In the figure of I. 9 prove that

1. the lines AF and DE are at right angles.

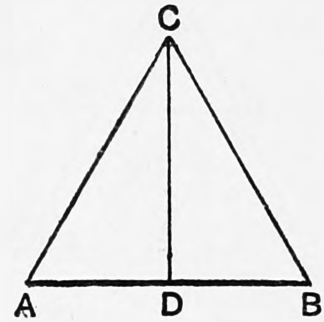
2. any point P in AF is equally distant from the points D and E.

3. Divide any angle into four equal parts.

Proposition 10. Problem.

To bisect a given finite straight line, that is, to divide it into two equal parts.

Let AB be the given straight line:
it is required to divide it into two equal parts.



Construction. On AB describe an equilateral triangle ABC, [I. 1.
and bisect the angle ACB by the straight line CD, which meets AB at D. [I. 9.
AB shall be cut into two equal parts at the point D.

Proof. In the triangles ACD, BCD,

because AC is equal to CB, [*Definition* 23.
and CD is common to both,
and the angle ACD is equal to the angle BCD, [*Construction.*

therefore the base AD is equal to the base DB. [I. 4.

Wherefore *the given straight line* AB *is divided into two equal parts at the point* D. [Q.E.F.

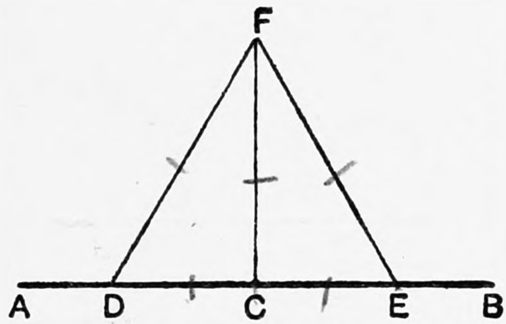
EXERCISE.

In the above figure prove that every point in the line CD, or CD produced, is equally distant from A and B.

PROPOSITION 11. PROBLEM.

To draw a straight line at right angles to a given straight line, from a given point in the same.

Let AB be the given straight line, and C the given point in it:
it is required to draw from the point C *a straight line at right angles to* AB.



Construction. Take any point D in AC, and make CE equal to CD. [I. 3.
On DE describe the equilateral triangle DFE, [I. 1.
and join CF.
The straight line CF shall be the line required

Proof. In the triangles DCF, ECF,

because DC is equal to CE, [*Construction*.
and CF is common to both,
and the base DF is equal to the base EF, [*Def.* 23.
therefore the angle DCF is equal to the angle ECF, [I. 8.
and they are adjacent angles.
But when a straight line, standing on another straight line, makes the adjacent angles equal to one another, each of the angles is called a right angle; [*Definition* 10.
therefore each of the angles DCF, ECF is a right angle.

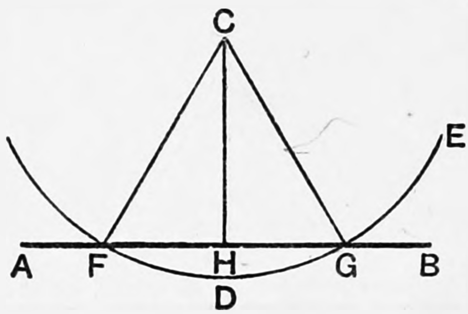
Wherefore, *from the given point* C *in the given straight line* AB, CF *has been drawn at right angles to* AB. [Q.E.F.

Proposition 12. Problem.

To draw a straight line perpendicular to a given straight line of an unlimited length from a given point without it.

Let AB be the given straight line, which may be produced to any length both ways, and let C be the given point without it:

it is required to draw from the point C *a straight line perpendicular to* AB.



Construction. Take any point D on the side of AB, remote from C, and with centre C and radius CD describe the circle EGF, meeting AB at F and G. [*Postulate* 3.

Bisect FG at H, [I. 10.

and join CH.

Then CH shall be the straight line required.

Join CF, CG.

Proof. In the triangles FHC, GHC,

because {FH is equal to HG, [*Construction.*
and HC is common to both,
and the base CF is equal to the base CG, [*Definition* 15.

therefore the angle CHF is equal to the angle CHG, [I. 8.

and they are adjacent angles.

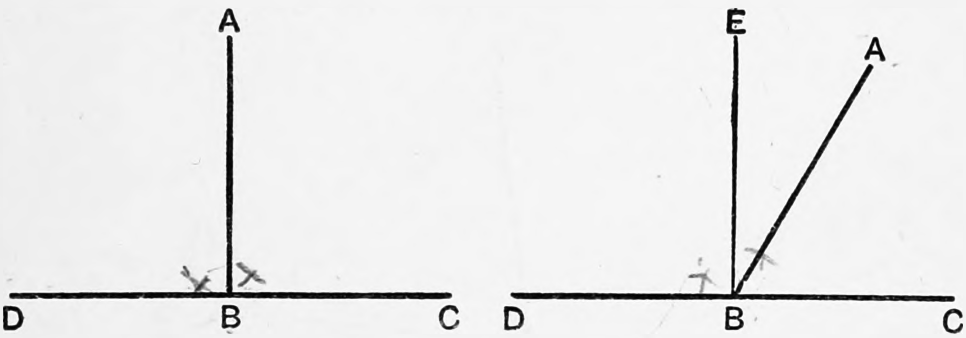
But when a straight line, standing on another straight line, makes the adjacent angles equal to one another, each of the angles is called a right angle, and the straight line which stands on the other is called a perpendicular to it. [*Def.* 10.

Wherefore *a perpendicular* CH *has been drawn to the given straight line* AB *from the given point* C *without it.* [Q.E.F.

Proposition 13. Theorem.

The angles which one straight line makes with another straight line on one side of it, are either two right angles, or are together equal to two right angles.

Let the straight line AB make with the straight line CD, on one side of it, the angles CBA, ABD:
these shall be either two right angles, or be together equal to two right angles.



Case I. If the angle CBA is equal to the angle ABD, each of them is a right angle. [*Definition* 10.

Case II. If not, from the point B draw BE at right angles to CD. [I. 11.

Proof. Since the angles DBE, EBC are equal to two right angles, [*Constr. and Def.* 10.
and the angle EBC is equal to the angles EBA, ABC,
therefore the angles DBE, EBA, ABC are equal to two right angles. [*Axiom* 2.
But the angles DBE, EBA are equal to the angle DBA;
therefore the angles DBA, ABC are equal to two right angles. [*Axiom* 2.

Wherefore, *the angles, etc.* [Q.E.D.

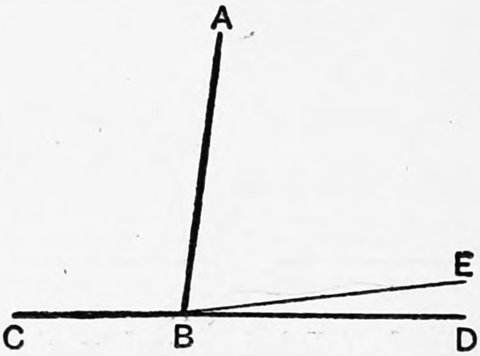
Note. Two angles, such as ABC and ABE, which together make up a right angle, EBC, are said to be **complementary**, and each is said to be the **complement** of the other.

Two angles, such as ABC and ABD, which together make up two right angles, are said to be **supplementary**, and each is said to be the **supplement** of the other.

PROPOSITION 14. THEOREM.

If, at a point in a straight line, two other straight lines, on the opposite sides of it, makes the adjacent angles together equal to two right angles, these two straight lines shall be in one and the same straight line.

At the point B in the straight line AB, let the two straight lines BC, BD, on the opposite sides of AB, make the adjacent angles ABC, ABD together equal to two right angles:
BD *shall be in the same straight line with* CB.



Proof. For if BD be not in the same straight line with CB, let BE be in the same straight line with it.
Then because the straight line AB meets the straight line CBE, the angles ABC, ABE are together equal to two right angles. [I. 13.
But the angles ABC, ABD are also together equal to two right angles. [*Hypothesis.*
Therefore the angles ABC, ABE are equal to the angles ABC, ABD. [*Axioms* 1 *and* 11.
From each of these equals take the common angle ABC, and the remaining angle ABE is equal to the remaining angle ABD, [*Axiom* 3.
the less to the greater, which is impossible.
Therefore BE is not in the same straight line with CB.

And in the same manner it may be shewn that no other can be in the same straight line with it but BD;
therefore BD is in the same straight line with CB.

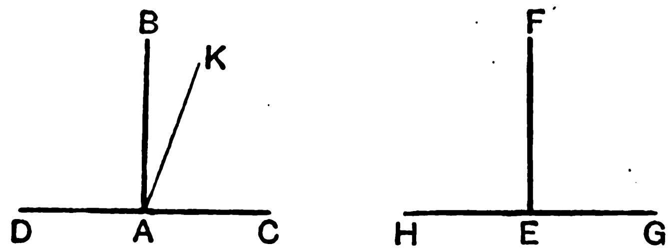
Wherefore, *if at a point, etc.* [Q.E.D.

NOTE TO I. 14.

Axiom 11, which is first used in this proposition, may be proved as follows:

Let AB be at right angles to DAC at the point A, and EF at right angles to HEG at the point E:

then shall the angles BAC *and* FEG *be equal.*



Take any length AC, and make AD, EH, EG all equal to AC. Now apply HEG to DAC, so that H may be on D, and HG on DC, and B and F on the same side of DC; then G will coincide with C, and E with A.

Also EF shall coincide with AB; for if not, suppose, if possible, that it takes a different position as AK.

Then the angle DAK is equal to the angle HEF, and the angle CAK to the angle GEF;

but the angles HEF and GEF are equal; [*Hyp. and Ax.* 11.

therefore the angles DAK and CAK are equal.

But the angles DAB and CAB are also equal, [*Hyp. and Ax.* 11.

and the angle CAB is greater than the angle CAK;

therefore the angle DAB is greater than the angle CAK.

Much more then is the angle DAK greater than the angle CAK.

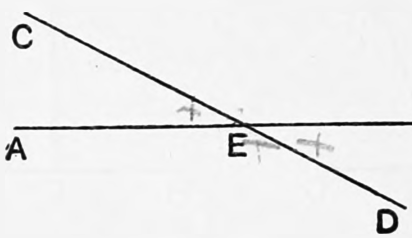
But the angle DAK was shewn to be equal to the angle CAK; which is absurd.

Therefore EF must coincide with AB; and therefore the angle FEG coincides with the angle BAC, and is equal to it.

PROPOSITION 15. THEOREM.

If two straight lines cut one another, the vertical, or opposite, angles shall be equal.

Let the two straight lines AB, CD cut one another at the point E:
the angle AEC *shall be equal to the angle* DEB,
and the angle CEB *to the angle* AED.



Proof. Because the straight line AE meets the straight line CD, the angles CEA, AED are together equal to two right angles. [I. 13.

Again, because DE meets AB, the angles AED, DEB are also together equal to two right angles. [I. 13.

Therefore the angles CEA, AED are equal to the angles AED, DEB. [*Axioms* 1 and 11.

From each of these equals take the common angle AED, and the remaining angle CEA is equal to the remaining angle DEB. [*Axiom* 3.

In the same manner it may be shewn that the angle CEB is equal to the angle AED.

Wherefore, *if two straight lines, etc.* [Q.E.D.

Corollary 1. From this it is clear that, if two straight lines cut one another, the angles which they make at the point where they cut are together equal to four right angles.

For the angles AEC, AED equal two right angles, and also the angles CEB, BED equal two right angles.

Corollary 2. It follows that all the angles made by any number of straight lines meeting at one point are together equal to four right angles.

EXERCISES ON PROPOSITION 12.

1. Prove that every point in CH, or CH produced, is equidistant from F and G.

2. Find a point in a given straight line such that its distances from two given points may be equal.

3. Find a point that shall be equidistant from three given points.

4. Through two given points on opposite sides of a given straight line draw two straight lines which shall meet in that given straight line, and include an angle bisected by that given straight line.

[Let A, B be the two points and KL the given straight line. Draw AM perpendicular to KL and produce to D, so that MD equals AM. Let DB meet KL in E; then AE, EB are the required lines.]

5. Through a given point draw a straight line which shall form with two given straight lines an isosceles triangle.

[Through the given point draw a straight line perpendicular to the bisector of the angle between the given straight lines.]

6. If the sides AB, AC of a triangle ABC be produced to D and E and the bisectors of the angles BCE, CBD meet in O, prove that the perpendiculars from O upon AD, AE, and BC are equal.

7. D is a given point in the base BC of a triangle ABC; find a straight line such that, if the triangle be folded along it, then the point A will coincide with D.

[The required straight line bisects AD at right angles.]

EXERCISES ON PROPOSITIONS 13-15.

1. If a right-angled triangle have one of its acute angles double the other, the hypotenuse is double the shorter side.

[If ABC be the right-angled triangle having the angle C double the angle B, produce CA to D, where CA=AD, and prove that CBD is equilateral, etc.]

2. If two isosceles triangles are on the same base, the straight line joining their vertices, or that straight line produced, will bisect the base at right angles.

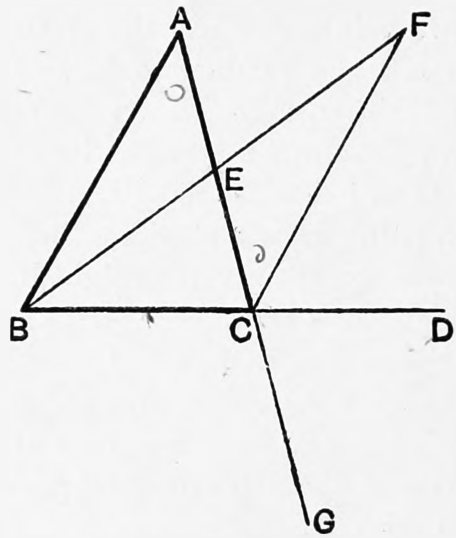
3. A given angle BAC is bisected; if CA is produced to G and the adjacent angle BAG bisected, the two bisecting lines are at right angles. [These two bisecting lines are called the interior and the exterior bisectors of the angles BAC.]

****4.** If four straight lines meet at a point so that the opposite angles are equal, these straight lines are two and two in the same straight line.

PROPOSITION 16. THEOREM.

If one side of a triangle be produced, the exterior angle shall be greater than either of the interior opposite angles.

Let ABC be a triangle, and let one side BC be produced to D:
the exterior angle ACD *shall be greater than either of the interior opposite angles* CBA, BAC.



Construction. Bisect AC at E, [I. 10.
join BE and produce it to F, making EF equal to EB, [I. 3.
and join FC.

Proof. In the triangles AEB, CEF,

because { AE = EC, [*Construction.*
and EB = EF, [*Construction.*
and the angle AEB = the angle CEF,

since they are opposite vertical angles; [I. 15.
therefore the angle BAE = the angle ECF. [I. 4.
But the angle ECD is greater than the angle ECF. [*Axiom* 8.
Therefore the angle ACD is greater than the angle BAE.

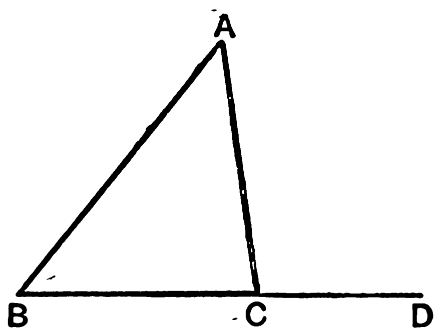
In the same manner, if BC be bisected, and the side AC be produced to G, it may be shewn that the angle BCG, that is, the angle ACD, is greater than the angle ABC. [I. 15.

Wherefore, *if one side, etc.* [Q.E.D.

PROPOSITION 17. THEOREM.

Any two angles of a triangle are together less than two right angles.

Let ABC be a triangle :
any two of its angles shall be together less than two right angles.



Construction. Produce BC to D.

Proof. Because ACD is the exterior angle of the triangle ABC, it is greater than the interior opposite angle ABC. [I. 16.
To each of these add the angle ACB.
Therefore the angles ACD, ACB are greater than the angles ABC, ACB.
But the angles ACD, ACB together = two right angles. [I. 13.
Therefore the angles ABC, ACB are together less than two right angles.

In the same manner it may be shewn that the angles BAC, ACB, as also the angles CAB, ABC, are together less than two right angles.

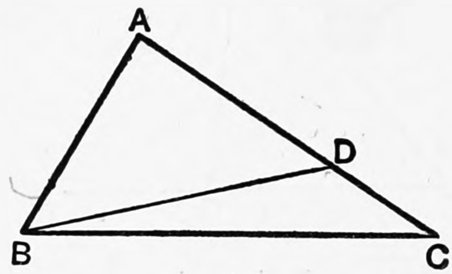
Wherefore, *any two angles, etc.* [Q.E.D.

Note. Every triangle has at least two acute angles; for if a triangle had two obtuse angles, their sum would not be less than two right angles, and this is impossible by the foregoing proposition.

PROPOSITION 18. THEOREM.

If one side of a triangle be greater than a second side, the angle opposite the first side shall be greater than the angle opposite the second.

Let ABC be a triangle, of which the side AC is greater than the side AB:
then shall the angle ABC *be also greater than the angle* ACB.



Construction. Because AC is greater than AB,
cut off AD equal to AB, [I. 3.
and join BD.

Proof. Because ADB is the exterior angle of the triangle BDC, it is greater than the interior opposite angle DCB. [I. 16.
But the angle ADB = the angle ABD, [I. 5.
because the side AD = the side AB. [*Construction.*
Therefore the angle ABD is also greater than the angle ACB.
Much more then is the angle ABC greater the angle ACB. [*Axiom* 8.

Wherefore, *the greater side, etc.* [Q.E.D.

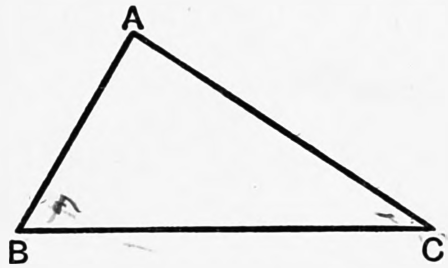
EXERCISE.

ABCD is a quadrilateral of which AD is the longest side and BC the shortest; shew that the angle ABC is greater than the angle ADC, and the angle BCD greater than the angle BAD.

PROPOSITION 19. THEOREM.

If one angle of a triangle be greater than a second angle, the side opposite the first angle shall be greater than the side opposite the second.

Let ABC be a triangle, of which the angle ABC is greater than the angle ACB:
then shall the side AC *be also greater than the side* AB.



Proof. For if not, AC must be either equal to AB or less than AB.

But AC is not equal to AB,
for then the angle ABC would be equal to the angle ACB; [I. 5.
but it is not; [*Hypothesis.*
therefore AC is not equal to AB.

Neither is AC less than AB,
for then the angle ABC would be less than the angle ACB; [I. 18.
but it is not; [*Hypothesis.*
therefore AC is not less than AB.
And it has been shewn that AC is not equal to AB.
Therefore AC is greater than AB.

Wherefore, *the greater angle, etc.* [Q. E. D.

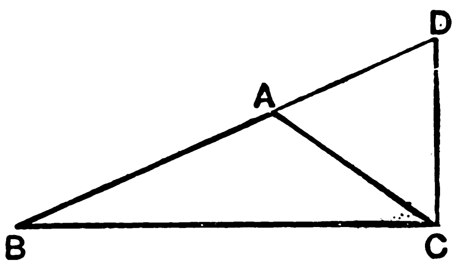
Note 1. In order to assist the student in remembering which of the two foregoing propositions is proved directly and which indirectly, it may be observed that the order is similar to that of Propositions 5 and 6.

Note 2. Simson's enunciations of Propositions 18 and 19 were:
the greater side of any triangle has the greater angle opposite it; and
the greater angle of any triangle is subtended by the greater side, that is, has the greater side opposite to it.

PROPOSITION 20. THEOREM.

Any two sides of a triangle are together greater than the third side.

Let ABC be a triangle :
any two sides of it are together greater than the third side,
namely, BA, AC *greater than* BC ;
AB, BC *greater than* AC ;
and BC, CA *greater than* AB.



Construction. Produce BA to D,
making AD equal to AC, [I. 3.
and join DC.

Proof. Because AD = AC, [*Construction.*
the angle ACD = the angle ADC. [I. 5.
But the angle BCD is greater than the angle ACD. [*Ax.* 8.
Therefore the angle BCD is greater than the angle BDC.
And because the angle BCD of the triangle BCD is greater than its angle BDC,
therefore the side BD is greater than the side BC. [I. 19.
But BD is equal to BA and AC.
Therefore BA and AC are together greater than BC.
In the same manner it may be shewn that
AB, BC are together greater than AC,
and BC, CA together greater than AB.
Wherefore, *any two sides, etc.* [Q.E.D.

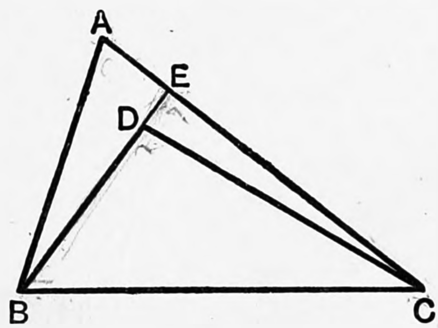
[For Exercises on Propositions 19 and 20 see page 39.]

PROPOSITION 21. THEOREM.

If from the ends of the side of a triangle there be drawn two straight lines to a point within the triangle, these shall be less than the other two sides of the triangle, but shall contain a greater angle.

Let ABC be a triangle, and from B and C, the ends of the side BC, let the two straight lines BD, CD be drawn to any point D within the triangle:

BD, DC *shall be less than the other two sides* BA, AC *of the triangle, but shall contain an angle* BDC *greater than the angle* BAC.



Construction. Produce BD to meet AC at E.

Proof. In the triangle ABE the two sides BA, AE are together greater than the side BE; [I. 20.

to each of these add EC;

∴ BA, AC are greater than BE, EC.

Again, the two sides DE, EC of the triangle DEC are greater than the third side DC; [I. 20.

to each of these add BD.

∴ BE, EC are greater than BD, DC.

But it has been shewn that BA, AC are greater than BE, EC; much more then are BA, AC greater than BD, DC.

Again, the exterior angle BDC of the triangle CDE is greater than the interior opposite angle CEB. [I. 16.

For the same reason, the exterior angle CEB of the triangle ABE is greater than the angle BAE;

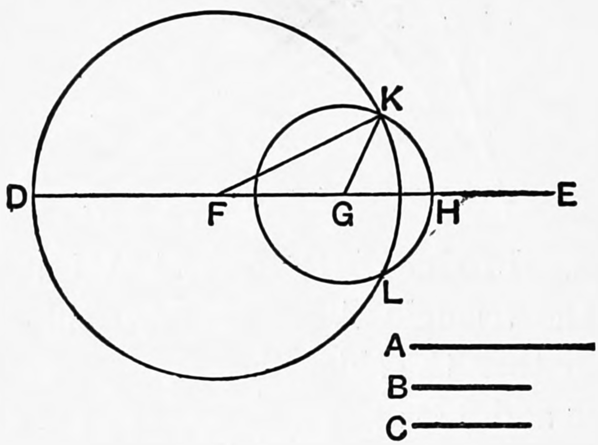
much more then is the angle BDC greater than the angle BAE, that is, BAC.

Wherefore, *if from the ends, etc.* [Q.E.D.

PROPOSITION 22. PROBLEM.

To make a triangle of which the sides shall be equal to three given straight lines, any two whatever of which are together greater than the third.

Let A, B, C be the three given straight lines, of which any two whatever are greater than the third; namely, A and B greater than C; A and C greater than B; and B and C greater than A:
it is required to make a triangle of which the sides shall be equal to A, B, C, *each to each.*



Construction. Take a straight line DE terminated at the point D, but unlimited towards E, and make DF equal to A, FG equal to B, and GH equal to C. [I. 3.
With centre F, and radius FD, describe the circle DKL. [*Postulate* 3.
With centre G, and radius GH, describe the circle HKL, and let it cut the former circle at K.
Join KF, KG.
The triangle KFG shall be drawn as required.

Proof. Because the point F is the centre of the circle DKL,

FD = FK. [*Definition* 15.

But FD = A; [*Construction.*

∴ FK = A. [*Axiom* 1.

Again, because the point G is the centre of the circle HLK,

GH = GK. [*Definition* 15.

But GH = C; [*Construction.*

∴ GK = C. [*Axiom* 1.

Also FG = B. [*Construction.*

Therefore the three straight lines KF, FG, GK are equal to the three A, B, C.

Wherefore *the triangle* KFG *has its three sides* KF, FG, GK *equal to the three given straight lines* A, B, C. [Q.E.F.

EXERCISES ON PROPOSITION 19.

1. ABC is a triangle and the angle A is bisected by a straight line which meets BC at D; shew that BA is greater than BD, and CA greater than CD.

2. If a straight line be drawn through A one of the angular points of a square, cutting one of the opposite sides, and meeting the other produced at F, shew that AF is greater than the diagonal of the square.

3. Every straight line drawn from the vertex of a triangle to the base is less than the greater of the two sides, or than either of them if they be equal.

4. ABC is a triangle in which BA is greater than CA; the angle A is bisected by a straight line which meets BC at D; shew that BD is greater than CD.

[Take a point E on AB such that AE = AC; then the angle AED equals the angle ACB, and ED is equal to CD; ∴ the angle BED equals the exterior angle at C, and is thus greater than the angle ABC; ∴ BD is greater than DE, that is, than CD.]

****5.** The perpendicular is the shortest straight line that can be drawn from a given point to a given straight line; and, of others, that which is nearer to the perpendicular is less than the more remote; and two, and two only, straight lines, each equal to a given straight line, can be drawn from the given point to the given straight line, one on each side of the perpendicular.

EXERCISES ON PROPOSITION 20.

****1.** The difference between any two sides of a triangle is less than the third side.

****2.** The sum of the distances of any point from the three angles of a triangle is greater than half the sum of the sides of the triangle.

3. The two sides of a triangle are together greater than **twice the straight line drawn from the vertex to the middle point of the base.

[*N.B.*—The straight line drawn from any angular point to the middle point of the opposite side is called a **Median.**]

[Let D be the middle point of the base BC of the triangle ABC. Produce AD to E where DE = AD, and join BE, EC; prove EC = AB, etc.]

4. The sum of the diagonals of a quadrilateral is less than the sum of the straight lines drawn to its angular points from any point except the intersection of its diagonals.

5. The four sides of any quadrilateral are together greater than the two diagonals together, but are less than twice the sum of the diagonals.

6. If through the ends of the base of a triangle with unequal sides lines be drawn to any point in the bisector of the vertical angle their difference is less than the difference of the sides.

[Let ABC be the $\triangle$, AD the bisector of the angle A, and P any point on AD. Take a point E on AB such that AE = AC; then PC = PE. Now BP < BE, EP;

$\therefore$ BP − EP < BE, *i.e.* BP − CP < BA − AE, *i.e.* < BA − AC.]

7. If lines be drawn to any point in the bisector of the exterior angle of a triangle from the ends of its base, their sum is greater than the sum of the sides.

[Take a point E on BA produced such that AE equals AC, and proceed as in the last Example.]

EXERCISE ON PROPOSITION 21.

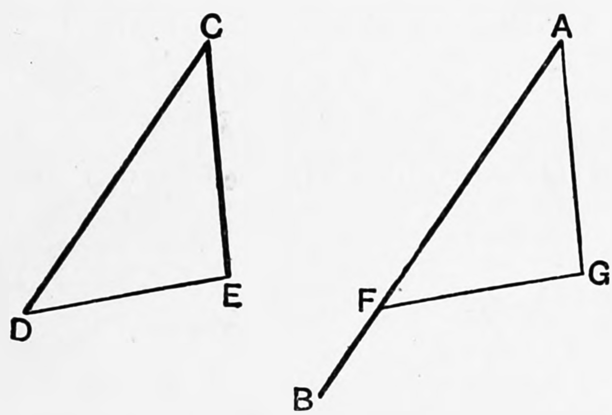
ABC is a triangle, and P is any point within it: shew that the sum of PA, PB, PC is less than the sum of the sides of the triangle.

PROPOSITION 23. PROBLEM.

At a given point in a given straight line, to make a rectilineal angle equal to a given rectilineal angle.

Let AB be the given straight line, and A the given point in it, and DCE the given rectilineal angle:
it is required to make at the given point A, *in the given straight line* AB, *an angle equal to the given rectilineal angle* DCE.

Construction. In CD, CE take any points D, E, and join DE.
From AB cut off AF equal to CD, and construct the triangle AFG so that the sides AF, FG, GA may be respectively equal to the sides CD, DE, EC. [I. 22.



The angle FAG shall be equal to the angle DCE.

Proof. In the triangles DCE, FAG,

because $\begin{cases} FA = DC, \\ \text{and } AG = CE, \\ \text{and the base } FG = \text{the base } DE; \end{cases}$ [*Construction.*

∴ the angle FAG = the angle DCE. [I. 8.

Wherefore, *at the given point* A, *in the given straight line* AB, *the angle* FAG *has been made equal to the given rectilineal angle* DCE. [Q.E.F.

[For Exercises see page 43.]

Proposition 24. Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides, equal to them, of the other, the base of that which has the greater angle shall be greater than the base of the other.

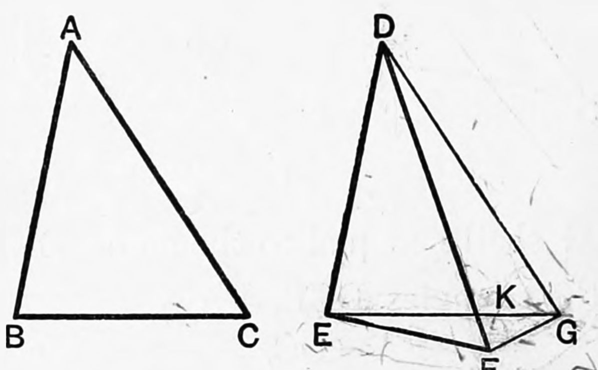
Let ABC, DEF be two triangles, which have the side AB equal to the side DE, the side AC equal to the side DF, but the angle BAC greater than the angle EDF :
the base BC *shall be greater than the base* EF.

Of the two sides DE, DF, let DE be the side which is not greater than the other.

Construction. At the point D in the straight line DE, make the angle EDG equal to the angle BAC, [I. 23.
and make DG equal to AC or DF ; [I. 3.

join EG, GF.

Let EG and DF, produced if necessary, meet in K.



Proof. Since DE is not greater than DF, that is, DG, that is, since DE is equal to, or less than, DG,
therefore the angle DGE is equal to, or less than, the angle DEG. [I. 5, 18.
But the angle DKG is greater than the angle DEG, [I. 16.
therefore the angle DKG is greater than the angle DGE ;
therefore the side DG is greater than DK,

that is, DF is greater than DK.

In the triangles ABC, DEG,

because $\begin{cases} AB = DE, & \text{[Hypothesis.} \\ \text{and } AC = DG, & \text{[Construction.} \\ \text{and the angle } BAC = \text{the angle } EDG, & \text{[Construction.} \end{cases}$

∴ the base BC = the base EG. [I. 4.

And because DG = DF, [*Construction.*

the angle DGF = the angle DFG. [I. 5.

But the angle DGF is greater than the angle EGF; [*Axiom* 8.

∴ the angle DFG is greater than the angle EGF.

Much more then is the angle EFG greater than the angle EGF. [*Axiom* 8.

And because the angle EFG of the triangle EFG is greater than its angle EGF,

∴ the side EG is greater than the side EF. [I. 19.

But EG was shown to be equal to BC;

∴ BC is greater than EF.

Wherefore, *if two triangles, etc.* [Q.E.D.

EXERCISES ON PROPOSITION 23.

1. If one angle of a triangle is equal to the sum of the other two, the triangle can be divided into two isosceles triangles.

2. If the angle C of a triangle is equal to the sum of the angles A and B, the side AB is equal to twice the straight line joining C to the middle point of AB.

3. Construct a triangle, having given the base, one of the angles at the base, and the sum of the sides.

[If AB be the given base, make the angle BAD equal to the given ∠ and AD equal to the given sum of the sides; at B, on same side of BD as A, make ∠DBK = ∠ADB; let BK meet AD in K. Then KAB is the required △.]

4. Construct a triangle having given the base, one of the base angles, and the difference of the sides.

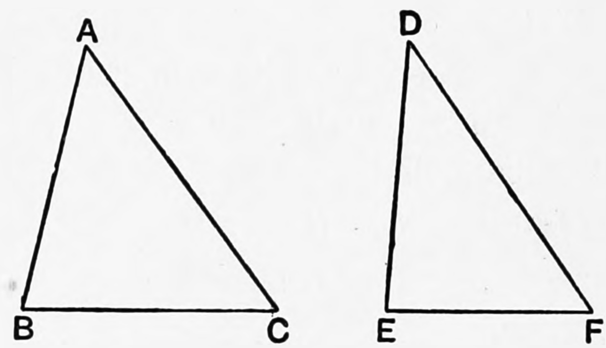
[If AB be the given base, make the angle BAD equal to the given ∠ and AD equal to the given difference; at B, on the other side of BD from A, make the ∠DBK = exterior ∠ of ADB; if BK meet AD produced in K, then KAB is the required △.]

5. A is a given point, and B is a given point in a given straight line: it is required to draw from A to the given straight line, a straight line AP, such that the sum of AP and PB may be equal to a given length.

Proposition 25. Theorem.

If two triangles have two sides of the one respectively equal to two sides of the other, but the base of the one greater than the base of the other, the angle contained by the sides of that which has the greater base shall be greater than the angle contained by the corresponding sides of the other.

Let ABC, DEF be two triangles, which have the side AB equal to the side DE, the side AC equal to the side DF, but the base BC greater than the base EF:
the angle BAC *shall be greater than the angle* EDF.



Proof. For if not, the angle BAC must be either equal to the angle EDF, or less than, the angle EDF.
But the angle BAC does not = the angle EDF, for then
the base BC would = the base EF; [I. 4.
but it is not; [*Hypothesis.*
∴ the angle BAC does not = the angle EDF.
Neither is the angle BAC less than the angle EDF,
for then the base BC would be less than the base EF; [I. 24.
but it is not; [*Hypothesis.*
∴ the angle BAC is not less than the angle EDF.
∴ the angle BAC is not equal, or less than, the angle EDF.
∴ the angle BAC must be greater than the angle EDF.

Wherefore, *if two triangles, etc.* [Q.E.D.

Note. Proposition 25, as well as Proposition 19, are proved by the method of "Exhaustion," that is, by shewing that no other conclusion, *except the given one*, can follow.

EXERCISES.

1. ABCD is a quadrilateral having AB, CD equal, but the diagonal BD greater than the diagonal AC; prove that the angle DCB is greater than the angle ABC.

2. ABC is a triangle having AB less than AC, and G is any point in the straight line joining A to the middle point of BC; prove that G is nearer to B than to C.

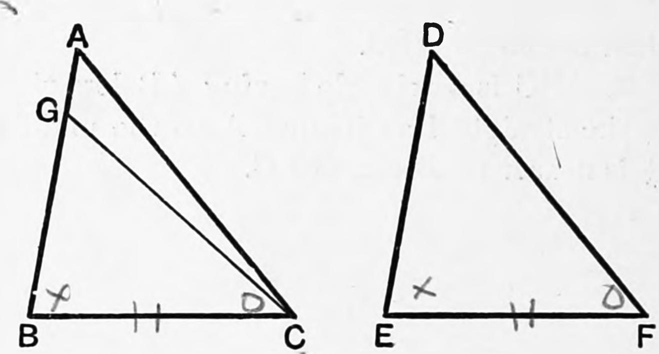
Proposition 26. Theorem.

If two triangles have two angles of the one respectively equal to two angles of the other, and one side equal to one side, namely, either the sides adjacent to the equal angles, or sides which are opposite to equal angles in each, then shall the triangles be equal in all respects, these sides being equal which are opposite to equal angles.

Case I. When the equal sides are adjacent to the equal angles.

Let ABC, DEF be two triangles. which have the angle ABC equal to DEF, and the angle BCA equal to EFD, and the side BC equal to the side EF,

then shall the triangles be equal in all respects, so that AB = DE, AC = DF, *and the angle* BAC = *the angle* EDF.



For if AB be not equal to DE, one of them must be greater than the other. Let AB be the greater, and make BG equal to ED, [I. 3. and join GC.

Then, in the triangles GBC, DEF,

because { GB = DE, [*Construction.*
and BC = EF,
and the angle GBC = the angle DEF, [*Hypothesis.*

∴ the triangles are equal in all respects,

and the angle GCB = the angle DFE. [I. 4.

But the angle DFE = the angle ACB. [*Hypothesis.*

∴ the angle GCB = the angle ACB, [*Axiom* 1.

the less to the greater, which is impossible.

Therefore AB is not unequal to DE, that is, it is equal to it.

Then, in the triangles ABC, DEF,

because { AB = DE, [*Proved.*
and BC = EF, [*Hypothesis.*
and the angle ABC = the angle DEF; [*Hypothesis.*

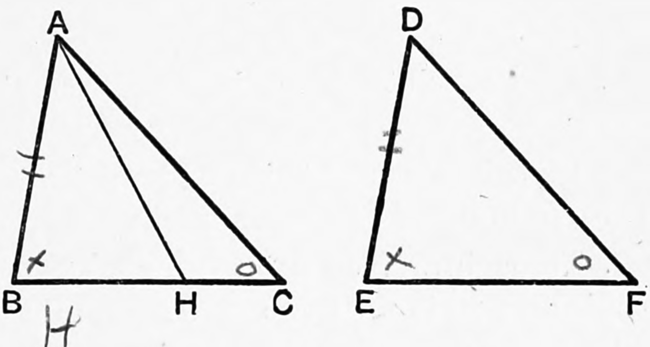
∴ the base AC = the base DF,

and the third angle BAC = the third angle EDF. [I. 4.

CASE II. When the equal sides are opposite to equal angles in each triangle.

Let ABC, DEF be two triangles having the two angles ABC, ACB respectively equal to the angles DEF, DFE, and the side AB equal to DE,

then the triangles shall be equal in all respects, so that BC = EF, AC = DF, *and also the third angle* BAC = *the third angle* EDF.



For if BC be not equal to EF, one of them must be greater than the other.

Let BC be the greater, and make BH equal to EF, [I. 3.

and join AH.

Then, in the triangles ABC, DEF,

because $\begin{cases} \text{AB} = \text{DE}, & \text{[\textit{Hypothesis}.} \\ \text{and BH} = \text{EF}, & \text{[\textit{Construction}.} \\ \text{and the angle ABH} = \text{the angle DEF}; & \text{[\textit{Hypothesis}.} \end{cases}$

∴ the triangle ABH = the triangle DEF in all respects,

and therefore the angle BHA = the angle EFD. [I. 4.

But the angle EFD = the angle BCA; [*Hypothesis.*

∴ the angle BHA = the angle BCA, [*Axiom* 1.

that is, the exterior angle BHA of the triangle AHC is equal to its interior opposite angle BCA, which is impossible. [I. 16.

Therefore BC is not unequal to EF, that is, it is equal to it.

Hence in the triangles ABC, DEF,

because $\begin{cases} \text{AB} = \text{DE}, & \text{[\textit{Hypothesis}.} \\ \text{and BC} = \text{EF}, & \text{[\textit{Proved}.} \\ \text{and the angle ABC} = \text{the angle DEF}, & \end{cases}$

∴ the base AC = the base DF,

and the third angle BAC = the third angle EDF. [I. 4.

Wherefore, *if two triangles, etc.* [Q.E.D.

EXERCISES ON PROPOSITION 26.

**1. The perpendiculars from the ends of the base of an isosceles triangle upon the opposite sides are equal.

**2. The perpendicular from the vertex on the base of an isosceles triangle bisects both the base and the vertical angle.

**3. The perpendiculars let fall on two given straight lines AB, AC from any point in the straight line bisecting the angle between them are equal.

4. Find a point whose distances from the sides of a given triangle are equal.

5. In a given straight line find a point such that the perpendiculars drawn from it to two given straight lines which intersect shall be equal.

6. Through a given point draw a straight line such that the perpendiculars on it from two given points may be on opposite sides of it and equal to each other.

7. A straight line bisects the angle A of a triangle ABC; from B a perpendicular is drawn to this bisecting straight line, meeting it at D, and BD is produced to meet AC or AC produced at E: shew that BD is equal to DE.

8. AB, AC are any two straight lines meeting at A: through any point P draw a straight line meeting them at E and F, such that AE may be equal to AF.

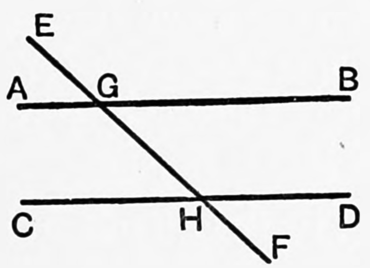
9. If the diagonal AC of a quadrilateral ABCD bisect the angles at A and C, prove that it is at right angles to the other diagonal BD.

Note to I. 26. The first twenty-six Propositions constitute a distinct section of the first Book of the *Elements.* The principal results are those contained in Propositions 4, 8, 26; in each of these Propositions it is shewn that two triangles which agree in three respects agree entirely. For another case in which two triangles are equal in all respects (in addition to the cases considered in I. 4, 8, 26) the student may refer to Page 322.

The Propositions from I. 27 to I. 34 inclusive may be said to constitute the second section of the first Book. They relate to the theory of parallel straight lines.

ON THE ANGLES MADE BY ONE STRAIGHT LINE WITH TWO OTHER STRAIGHT LINES.

When two straight lines AB and CD are met by a third straight line EF in two points G, H, names for the sake of distinction are given to the angles at G and H.



Thus EGB, AGE, CHF and FHD are called **exterior** angles;
BGH, AGH, CHG, GHD are called **interior** angles;
BGH and GHC are called **alternate** angles,
and so are also AGH and GHD;
also GHD is said to be *the interior and opposite angle of the exterior angle* EGB.

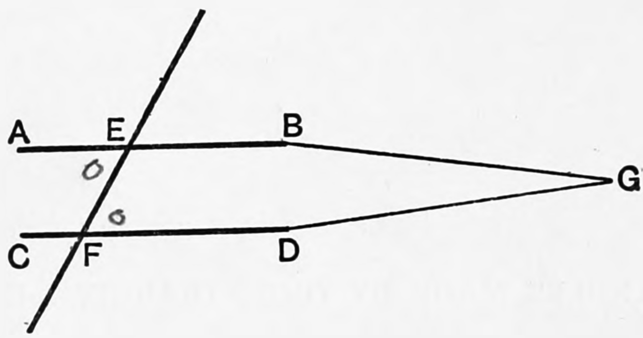
The words *interior*, *exterior*, and *alternate* are often abbreviated into *int.*, *ext.*, and *alt.*

PROPOSITION 27. THEOREM.

If a straight line, meeting two other straight lines, make the alternate angles equal to one another, the two straight lines shall be parallel to one another.

Let the straight line EF, which meets the two straight lines AB, CD, make the alternate angles AEF, EFD equal to one another:

AB *shall be parallel to* CD.



Proof. For if not, AB and CD, being produced, will meet either towards B, D or towards A, C. Let them be produced and meet towards B, D at the point G.

Therefore GEF is a triangle, and its exterior angle AEF is greater than the interior opposite angle EFG. [I. 16.

But the angle AEF also = the angle EFG; [*Hypothesis.*

which is impossible.

∴ AB and CD, being produced, do not meet towards B, D.

In the same manner it may be shewn that they do not meet towards A, C.

∴ AB is parallel to CD. [*Definition* 29.

Wherefore, *if a straight line, etc.* [Q.E.D.

PROPOSITION 28. THEOREM.

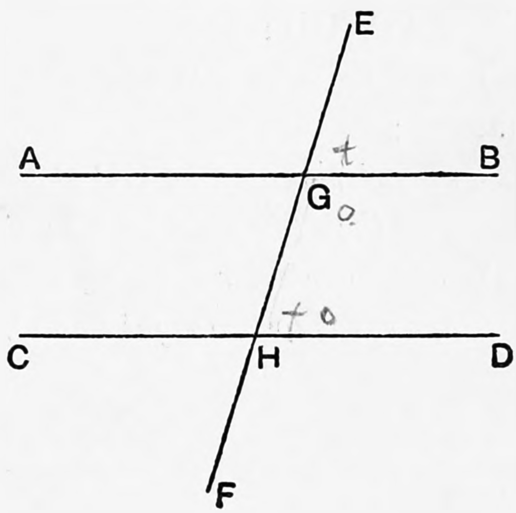
If a straight line, meeting two other straight lines, make the exterior angle equal to the interior and opposite angle on the same side of the line, or if it make the interior angles on the same side together equal to two right angles, the two straight lines shall be parallel to one another.

Let the straight line EF, which meets the two straight lines AB, CD,

(1) make the exterior angle EGB equal to the interior and opposite angle GHD on the same side;

or (2) make the interior angles on the same side BGH, GHD together equal to two right angles:

AB *shall be parallel to* CD.



Proof. (1) Because the angle EGB = the angle GHD, [*Hyp.*
and the angle EGB = the angle AGH, [I. 15.
∴ the angle AGH = the angle GHD, [*Axiom* 1.
and they are alternate;
∴ AB is parallel to CD. [I. 27.

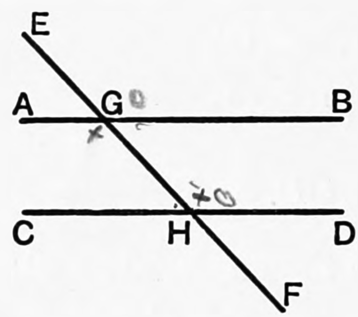
(2) Again, because the angles BGH, GHD are together equal to two right angles, [*Hypothesis.*
and the angles AGH, BGH together = two right angles, [I. 13.
∴ the angles AGH, BGH = the angles BGH, GHD. [*Axs.* 1, 11.
Take away the common angle BGH;
∴ the angle AGH = the angle GHD, [*Axiom* 3.
and they are alternate;
∴ AB is parallel to CD. [I. 27.

Wherefore, *if a straight line, etc.* [Q.E.D.

PROPOSITION 29. THEOREM.

If a straight line meet two parallel straight lines, it makes
(1) *the alternate angles equal to one another;*
(2) *the exterior angle equal to the interior and opposite angle on the same side;*
and also (3) *the two interior angles on the same side together equal to two right angles.*

Let the straight line EF meet the two parallel straight lines AB, CD:
(1) *then the alternate angles* AGH, GHD *shall be equal to one another;*
(2) *the exterior angle* EGB *shall be equal to the interior and opposite angle on the same side* GHD; *and*
(3) *the two interior angles on the same side* BGH, GHD *shall be together equal to two right angles.*



Proof. (1) If the angle AGH be not equal to the angle GHD, one of them must be greater than the other; let the angle AGH be the greater.
Then the angle AGH is greater than the angle GHD;
to each of them add the angle BGH;
∴ the angles AGH, BGH are greater than the angles BGH, GHD.
But the angles AGH, BGH together = two right angles; [I. 13.
∴ the angles BGH, GHD are together less than two right angles;
∴ AB, CD, if continually produced, will meet on the side of GH towards B and D. [*Axiom* 12.
But they never meet, since they are parallel by hypothesis.

∴ the angle AGH is not unequal to the angle GHD, that is, it is equal to it.

(2) Again, because the angle AGH = the angle EGB; [I. 15.
therefore the angle EGB = the angle GHD. [*Axiom* 1.

(3) Add to each of these the angle BGH.
∴ the angles EGB, BGH = the angles BGH, GHD. [*Axiom* 2.
But the angles EGB, BGH together = two right angles. [I. 13.
∴ the angles BGH, GHD together = two right angles. [*Ax.* 1.

Wherefore, *if a straight line, etc.* [Q.E.D.

[For Exercises see page 55.]

NOTE ON EUCLID'S TWELFTH AXIOM.

In I. 29 Euclid uses for the first time his twelfth axiom. The theory of parallel lines has always been considered the great difficulty of elementary geometry, and many attempts have been made to overcome this difficulty in a better way than Euclid has done. We shall not give an account of these attempts.

Speaking generally, it may be said that the methods which differ substantially from Euclid's involve, in the first place, an axiom as difficult as his, and then an intricate series of propositions; while in Euclid's method, after the axiom is once admitted, the remaining process is simple and clear.

One modification of Euclid's axiom has been proposed, which appears to diminish the difficulty of the subject. This consists in assuming, instead of Euclid's axiom, the following:

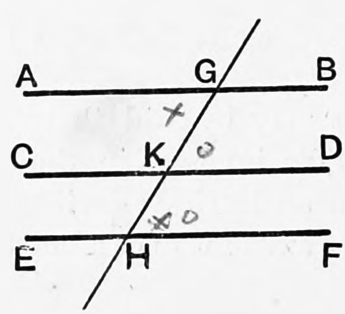
Two intersecting straight lines cannot be both parallel to a third straight line. The propositions in the *Elements* are then demonstrated as in Euclid up to I. 28, inclusive. Then, in I. 29, we proceed with Euclid up to the words, "therefore the angles BGH, GHD are together less than two right angles." We then infer that BGA and CHD must meet: because if a straight line be drawn through G so as to make the interior angles together equal to two right angles, this straight line will be parallel to CD, by I. 28; and, by our axiom, there cannot be two parallels to CD, both passing through G.

This form of making the necessary assumption has been recommended by various eminent mathematicians, among whom may be mentioned Playfair and De Morgan. It is hence known as **Playfair's Axiom.** By postponing the consideration of the axiom until it is wanted, that is, until after I. 28, and then presenting it in the form here given, the theory of parallel straight lines appears to be treated in the easiest manner that has hitherto been proposed.

PROPOSITION 30. THEOREM.

Straight lines which are parallel to the same straight lines are parallel to each other.

Let AB, CD be each of them parallel to EF:
then shall AB *be parallel to* CD.



Construction. Let the straight line GKH cut AB, EF, CD.

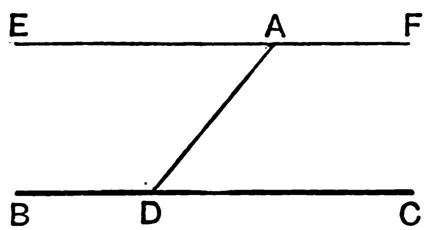
Proof. Because GKH cuts the parallel lines AB, EF, the angle AGH = the alternate angle GHF. [I. 29.
Again, because GK cuts the parallel lines EF, CD, the exterior angle GKD = the interior opposite angle GHF. [I. 29.
∴ the angle AGK = the angle GKD, [*Axiom* 1.
and they are alternate angles;
∴ AB is parallel to CD. [I. 27.

Wherefore, *straight lines, etc.* [Q.E.D.

PROPOSITION 31. PROBLEM.

To draw a straight line through a given point parallel to a given straight line.

Let A be the given point, and BC the given straight line: *it is required to draw a straight line through the point* A *parallel to the straight line* BC.



Construction. In BC take any point D, and join AD; at the point A in the straight line AD make the angle DAE equal to the alternate angle ADC, [I. 23.

and produce EA to F.

EF shall be parallel to BC.

Proof. Because AD, which meets the two straight lines BC, EF, makes the alternate angles EAD, ADC equal. [*Constr.*

∴ EF is parallel to BC. [I. 27.

Wherefore *the straight line* EAF *is drawn through the given point* A *parallel to the given straight line* BC. [Q.E.F.

EXERCISES ON PROPOSITION 29.

1. In the figure of I. 16 prove that AB and FC are parallel.

****2.** Straight lines which are perpendicular to the same straight line are parallel.

3. Any straight line parallel to the base of an isosceles triangle makes equal angles with the sides.

4. If two straight lines A and B are respectively parallel to two others C and D, shew that the inclination of A to B is equal to that of C to D.

5. A straight line is drawn terminated by two parallel straight lines; through its middle point any straight line is drawn and terminated by the parallel straight lines. Shew that the second straight line is bisected at the middle point of the first.

**6. If through any point equidistant from two parallel straight lines, two straight lines be drawn cutting the parallel straight lines, they will intercept equal portions of these parallel straight lines.

7. If the straight line bisecting the exterior angle of a triangle be parallel to the base, shew that the triangle is isosceles.

EXERCISES ON PROPOSITION 31.

1. Place between two parallel straight lines, and terminated by them, a straight line of given length.

2. Find a point B in a given straight line CD, such that if AB be drawn to B from a given point A, the angle ABC will be equal to a given angle.

[Draw AE parallel to CD and make the angle EAK equal to the given angle, AK being on the same side of AE as CD; AK meets CD in the required point B.]

3. Find a point such that the perpendiculars from it upon two given straight lines may be given. How many such points are there?

4. From a point D in the base BC of an isosceles triangle ABC a straight line DEF is drawn perpendicular to BC to meet the sides in E and F; prove that AEF is an isosceles triangle.

5. Through the middle point M of the base BC of a triangle a straight line DME is drawn, so as to cut off equal parts from the sides AB, AC, produced if necessary; show that BD is equal to CE.

[Through C draw CF parallel to AB to meet DE in F; prove that CF=BD, and also that CF=CE.]

6. Construct a triangle, having given the base, the altitude, and the length of the line joining the vertex to the middle point of the base.

7. Find points D, E in the sides AB, AC of a triangle ABC, such that DE is parallel to BC and equal to BD.

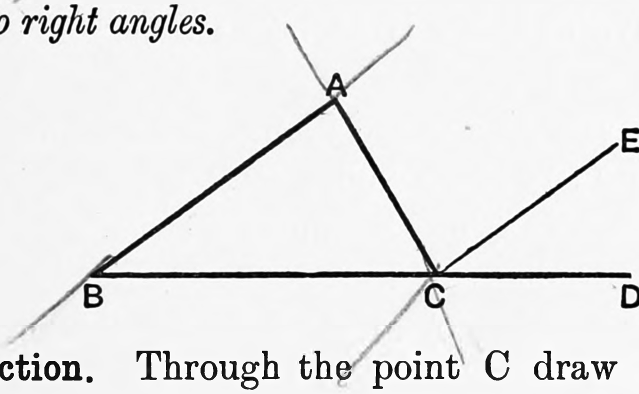
[BE bisects the angle ABC.]

8. Given the altitude and the base angles of a triangle, construct it.

PROPOSITION 32. THEOREM.

If a side of any triangle be produced,
(1) *the exterior angle is equal to the two interior and opposite angles;* and (2) *the three interior angles of the triangle are together equal to two right angles.*

Let ABC be a triangle, and let one of its sides BC be produced to D:
then shall (1) *the exterior angle* ACD *be equal to the two interior and opposite angles* CAB, ABC;
and (2) *the three interior angles* ABC, BCA, CAB *shall be together equal to two right angles.*



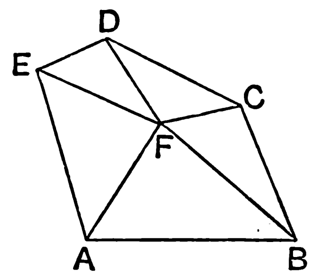
Construction. Through the point C draw CE parallel to AB. [I. 31.

Proof. (1) Because AB is parallel to CE, and AC meets them, the alternate angles BAC, ACE are equal. [I. 29.
Again, because AB is parallel to CE, and BD meets them, the exterior angle ECD is equal to the interior and opposite angle ABC. [I. 29.
∴ the whole exterior angle ACD = the two interior and opposite angles CAB, ABC. [*Axiom* 2.

(2) To each of these equals add the angle ACB;
∴ the angles ACD, ACB = the three angles CBA, BAC, ACB. [*Axiom* 2.
But the angles ACD, ACB together = two right angles; [I. 13.
∴ also the angles CBA, BAC, ACB together = two right angles. [*Axiom* 1.

Wherefore, *if a side of any triangle, etc.* [Q.E.D.

Corollary 1. *All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has sides.*



Take any rectilineal figure, and join any point F within it to its angular points.

We thus have as many triangles as there are sides to the figure.

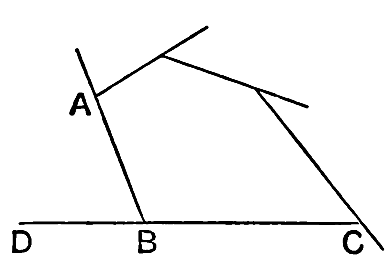
Also, by the preceding propositions, the three angles of each triangle make up two right angles.

∴ all the angles of these triangles = twice as many right angles as the figure has sides.

But all the angles of the triangles = the interior angles of the figure, together with the angles at the point F, which are equal to four right angles. [I. 15. *Cor.* 2.

∴ all the interior angles of the figure, together with four right angles, = twice as many right angles as the figure has sides.

Corollary 2. *All the exterior angles of any rectilineal figure are together equal to four right angles.*



Because every interior angle ABC, with its adjacent exterior angle ABD, is equal to two right angles; [I. 13.

∴ all the interior angles of the figure, together with all its

exterior angles, = twice as many right angles as the figure has sides.

But all the interior angles of the figure, together with four right angles, = twice as many right angles as the figure has sides. [*Corollary* 1.

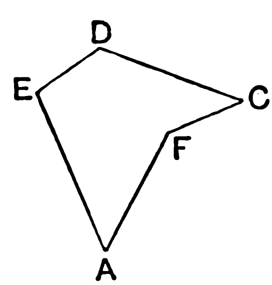
∴ all the interior angles of the figure, together with all its exterior angles, = all the interior angles of the figure, together with four right angles.

∴ all the exterior angles = four right angles.

NOTE ON THE COROLLARIES TO I. 32.

I. 32. The corollaries to I. 32 were added by Simson. In the second corollary it ought to be stated what is meant by an *exterior* angle of a rectilineal figure. At each angular point let *one* of the sides meeting at that point be produced; then the exterior angle at that point is the angle contained between this produced part and the side which is not produced. *Either* of the sides may be produced, for the two angles which can thus be obtained are equal, by I. 15.

The rectilineal figures to which Euclid confines himself are those in which the angles all face inwards; we may here however notice another class of figures. In the accompanying diagram the angle AFC faces



outwards, and it is an angle less than two right angles; this angle however is not one of the interior angles of the figure AEDCF. We may consider the corresponding interior angle to be the excess of four right angles above the angle AFC.

An angle such as AFC, greater than two right angles, is called a **re-entrant angle**.

The *first* of the corollaries to I. 32 is true for a figure which has a re-entrant angle or re-entrant angles; but the *second* is not.

EXERCISES.

****1.** If two triangles have two angles of the one equal to two angles of the other, the third angle of the one is equal to the third angle of the other.

****2.** Each angle of an equilateral triangle is equal to two-thirds of a right angle. Hence trisect a right angle.

****3.** The sum of the angles of any quadrilateral figure is equal to four right angles.

****4.** If one angle of a triangle be equal to the sum of the other two, the triangle is right-angled ; if it be less than the sum of the other two it is acute ; if greater, then obtuse.

5. In an obtuse-angled triangle the side opposite the obtuse angle is the greatest side.

6. What is the magnitude of an angle of a regular (1) pentagon, (2) hexagon, and (3) octagon ?

7. The bisectors of the exterior angles of a quadrilateral form a quadrilateral, the sum of whose two opposite angles is two right angles.

****8.** The straight line joining the middle point of the hypotenuse of a right-angled triangle to the right angle is equal to half the hypotenuse.

[A being the right ∠ of the △ABC, make ∠BAD=∠ABC, and let AD meet BC in D ; then ∠DAC=∠DCA, etc.]

9. From the extremities of the base of an isosceles triangle straight lines are drawn perpendicular to the sides ; shew that the angles made by them with the base are each equal to half the vertical angle.

10. If the straight lines bisecting the angles at the base of an isosceles triangle be produced to meet, they will contain an angle equal to an exterior angle of the triangle.

11. ABC is a triangle, and the exterior angles at B and C are bisected by the straight lines BD, CD respectively, meeting at D ; shew that the angle BDC together with half the angle BAC make up a right angle.

12. The angle between the internal bisector of one base angle and the external bisector of the other is equal to one-half the vertical angle.

13. The angle included between the bisector of the angle A of a triangle and the perpendicular from A upon the opposite side is half the difference of the base angles of the triangle.

14. The bisector of the exterior vertical angle of a triangle makes with the base an angle equal to half the difference of the base angles, and with either side an angle equal to half the sum of the base angles.

15. On the sides of any triangle ABC equilateral triangles BCD, CAE, ABF are described, all external; shew that the straight lines AD, BE, CF are all equal.

16. Through two given points draw two straight lines forming with a straight line given in position an equilateral triangle.

[Through the points draw straight lines making with the given line angles equal to two-thirds of a right angle.]

17. A is the vertex of an isosceles triangle ABC, and BA is produced to D, so that AD is equal to BA, and DC is drawn ; shew that BCD is a right angle.

18. The median passing through the vertex of a triangle is equal to, greater than, or less than half of the base according as the vertical angle is a right, an acute, or an obtuse angle. [Use Ex. 4.]

19. If one angle of a triangle be triple another the triangle may be divided into two isosceles triangles.

[Let ABC be the $\triangle$ where $\angle BCA = 3\angle ABC$; at C make $\angle BCD = \angle ABC$, and let CD meet AB in D. Then DBC, ADC can be proved to be isosceles triangles.]

20. If one angle of a triangle be double another, an isosceles triangle may be added to it so as to form together with it a single isosceles triangle.

[Let ABC be the $\triangle$ where $\angle BCA = 2\angle ABC$; with centre A and radius AB describe a circle meeting BC produced in D ; then ACD will be the $\triangle$ to be added.]

21. Given two angles of a triangle and the side opposite one of them, construct the triangle.

22. Within a triangle ABC straight lines AD, BE, and CF are drawn, making with AB, BC, and CA respectively the angles DAB, EBC, FCA equal to each other. If AD, BE, and CF do not meet in one point they will form by their intersections a triangle whose angles are equal to those of the triangle ABC.

23. Through the vertex A of a triangle ABC a straight line is drawn parallel to BC. If, on either side of A, lengths AD, AE be measured off from it, each equal to AC, and CD, CE be joined, prove that DCE is a right angle.

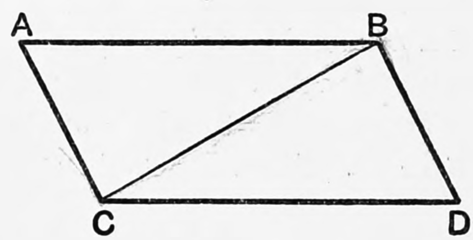
****24.** Each of the angles of a polygon of n sides, whose angles are all equal, is equal to $\frac{2n-4}{n}$ right angles.

PROPOSITION 33. THEOREM.

The straight lines which join the extremities of two equal and parallel straight lines towards the same parts are also themselves equal and parallel.

Let AB and CD be equal and parallel straight lines, and let them be joined towards the same parts by the straight lines AC and BD:

then shall AC *and* BD *be equal and parallel.*



Construction. Join BC.

Proof. Because AB is parallel to CD, [*Hypothesis.*
and BC meets them,
the alternate angles ABC, BCD are equal. [I. 29.

Then, in the triangles ABC, BCD,

because { AB = CD, [*Hypothesis.*
and BC is common,
and the angle ABC = the angle BCD; [*Proved.*

∴ the triangles are equal in all respects, so that
the base AC = the base BD,
and the angle ACB = the angle CBD. [I. 4.

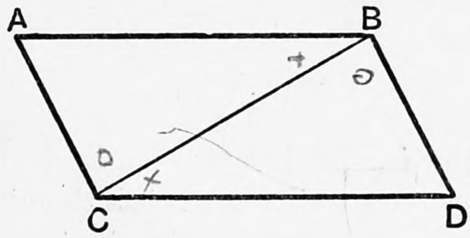
Also these are alternate angles;
∴ AC is parallel to BD, [I. 27.
and it was shewn to be equal to it.

Wherefore, *the straight lines, etc.* [Q.E.D.

PROPOSITION 34. THEOREM.

The opposite sides and angles of a parallelogram are equal to one another, and the diagonal bisects the parallelogram, that is, divides it into two equal parts.

Let ACDB be a parallelogram, of which BC is a diagonal: *then* (1) *the opposite sides and angles of the figure shall be equal to one another, and* (2) *the diagonal* BC *shall bisect it.*



Proof. (1) Because AB is parallel to CD, and BC meets them, the alternate angles ABC, BCD are equal. [I. 29.

And because AC is parallel to BD, and BC meets them, the alternate angles ACB, CBD are equal. [I. 29.

Then, in the triangles ABC, DCB,

because $\begin{cases} \text{the angle ABC = angle DCB,} \\ \text{and the angle BCA = angle CBD,} \\ \text{and the side BC adjacent to the equal angles in each} \\ \text{is common to both;} \end{cases}$

∴ the triangles are equal in all respects, so that

$\begin{cases} \text{AB = CD,} \\ \text{AC = BD,} \\ \text{and the angle BAC = the angle CDB.} \end{cases}$ [I. 26.

And because the angle ABC = the angle BCD,
and the angle CBD = the angle ACB,
∴ the whole angle ABD = the whole angle ACD. [*Ax.* 2.

And the angle BAC has been proved equal to the angle CDB.

∴ the opposite sides and angles are equal.

(2) Also it has been proved that the triangles ABC, DCB are equal in all respects;

∴ the diagonal BC bisects the parallelogram ACDB.

Wherefore, *the opposite sides, etc.* [Q.E.D.

EXERCISES.

1. If a quadrilateral have two of its opposite sides parallel, and the two others equal but not parallel, any two of its opposite angles are together equal to two right angles.

****2.** If the opposite angles of a quadrilateral are equal it is a parallelogram. [$A+B=C+D=$two rt. $\angle^s$, by I. 32, Cor. 2; $\therefore$ etc.]

****3.** If the opposite sides of a quadrilateral are equal it is a parallelogram.

****4.** The diagonals of a parallelogram bisect each other.

****5.** If the diagonals of a quadrilateral bisect each other it is a parallelogram.

****6.** If the straight line joining two opposite angles of a parallelogram bisect the angles the parallelogram is a rhombus.

7. Draw a straight line through a given point such that the part of it intercepted between two given parallel straight lines may be of given length. [Use Ex. 1, page 56.]

****8.** Straight lines bisecting two adjacent angles of a parallelogram intersect at right angles.

9. Straight lines bisecting two opposite angles of a parallelogram are either parallel or coincident.

****10.** If the diagonals of a parallelogram are equal all its angles are equal, and it is a rectangle.

11. Shew that any straight line passing through the middle point of the diagonal of a parallelogram and terminated by two opposite sides, bisects the parallelogram.

12. Bisect a parallelogram by a straight line drawn through any given point. [Use Ex. 11.]

****13.** The diagonals of a rhombus are at right angles.

14. A, B, C are three points in a straight line, such that AB is equal to BC; shew that the sum of the perpendiculars from A and C on any straight line which does not pass between A and C is double the perpendicular from B on the same straight line.

15. If straight lines be drawn from the angles of any parallelogram perpendicular to any straight line which is outside the parallelogram, the sum of those from one pair of opposite angles is eqnal to the sum of those from the other pair of opposite angles. [Use Ex. 14.]

****16.** The parallel to any side of a triangle through the middle point of another bisects the third side. [See Appendix, Art. 1.]

17. On the sides of a parallelogram ABCD four points E, F, G and H are taken, and parallels to the sides are drawn through these points

to form a parallelogram PQRS. Prove that the sum of the areas ABCD and PQRS is equal to twice that of EFGH.

[Twice EFGH = twice PQRS + $\|^{gms}$ PB, QC, RD, SA = etc.]

18. In the base BC of an isosceles triangle ABC any point D is taken, and perpendiculars are drawn from D upon the two sides AB, AC. Prove that the sum of these two perpendiculars is equal to the perpendicular from B upon AC.

[Let DE, DF be the perpendiculars, and draw BK perp^r to AC and DL perp^r to BK; prove the two $\triangle^s$ BED, DLB equal in all respects and $\therefore$ BL = DE, etc.]

19. From a given point O draw to a given straight line AB a straight line that shall be bisected by another given straight line AC.

[Draw OP $\|^l$ to AB to meet AC in P, and PQ $\|^l$ to OA to meet AB in Q; then OQ is the required straight line, by Ex. 4.]

20. It is required to draw a straight line which shall be equal to one straight line and parallel to another, and be terminated by two given straight lines.

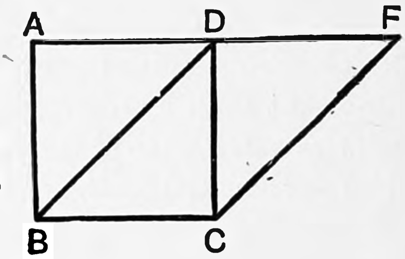
****21.** AB, AC are two given straight lines; through a given point E between them it is required to draw a straight line GEH such that the intercepted portion GH shall be bisected at the point E. [See Appendix, Art. 18.]

Note.—The Propositions from I. 35 to I. 48 may be said to constitute the third section of the first Book of the *Elements*. They relate to equality of area in figures which are not necessarily identical in form.

Proposition 35. Theorem.

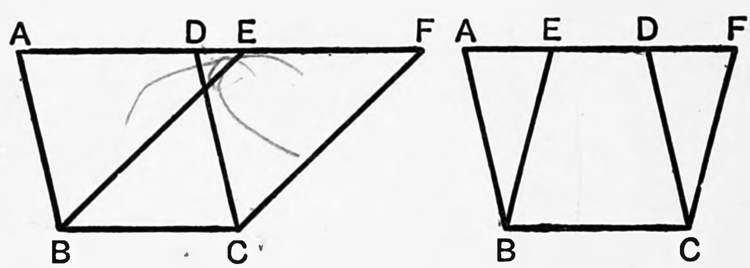
Parallelograms on the same base, and between the same parallels, are equal to one another.

Let the parallelograms ABCD, EBCF be on the same base BC, and between the same parallels AF, BC:
then shall the parallelogram ABCD *be equal to the parallelogram* EBCF.



Proof. Case I. If the sides AD, DF opposite to the base BC be terminated at the same point D, it is plain that each of the parallelograms is double of the triangle BDC; [I. 34.
and they are therefore equal. [*Ax.* 6.

Case II. But if the sides AD, EF, opposite to the base BC be not terminated at the same point, then
because ABCD is a parallelogram, AD = BC; [I. 34.



for the same reason EF = BC; ∴ AD = EF; [*Axiom* 1.
∴ the whole, or the remainder, AE = the whole, or the remainder, DF. [*Axioms* 2, 3.

Then, in the triangles EAB, FDC,

because $\begin{cases} \text{AB} = \text{DC}, \\ \text{and AE} = \text{DF}, \\ \text{and the exterior angle FDC} = \text{the interior opposite} \\ \text{angle EAB;} \end{cases}$ [I. 29.

∴ the △EAB = the △FDC. [I. 4.

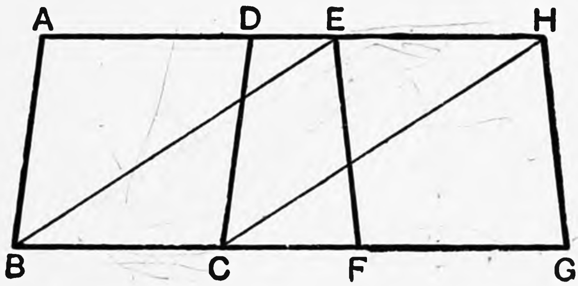
Take the △FDC from the figure ABCF, and from the same, or a similar, figure take the △EAB, and the remainders are equal; [*Axiom* 3.
that is, the parallelogram ABCD = the parallelogram EBCF.

Wherefore, *parallelograms on the same base, etc.* [Q.E.D.

PROPOSITION 36. THEOREM.

Parallelograms on equal bases, and between the same parallels, are equal to one another,

Let ABCD, EFGH be parallelograms on equal bases BC, FG, and between the same parallels AH, BG:
then shall ABCD *be equal to* EFGH.



Construction. Join BE, CH.

Proof. Because BC = FG, [*Hypothesis.*
and FG = EH, [I. 34.
∴ BC = EH, [*Axiom* 1.
and they are parallels; [*Hypothesis.*
∴ BE and CH, which join their extremities toward the same parts, are both equal and parallel. [I. 33.
∴ EBCH is a parallelogram, [*Definition.*
and it is equal to ABCD, because they are on the same base BC, and between the same parallels BC, AH. [I. 35.

Also the parallelogram EFGH is equal to the same EBCH, since they are on the same base EH, and between the same parallels EH, BG. [I. 35.
∴ the parallelogram ABCD = the parallelogram EFGH. [*Ax.* 1.
Wherefore, *parallelograms, etc.* [Q.E.D.

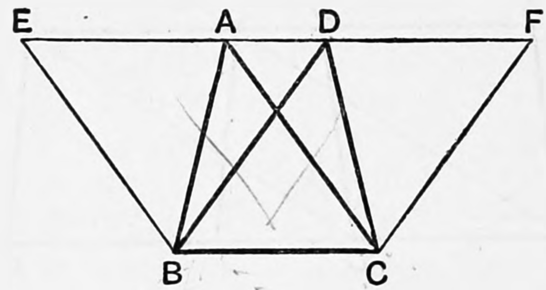
EXERCISES.

1. Construct a rectangle equal to a given parallelogram.
2. Construct a rhombus equal to a given parallelogram.
3. Divide a parallelogram into four equal parallelograms.
4. If two adjacent sides of a parallelogram be equal, its area is greatest when the sides are perpendicular.

Proposition 37. Theorem.

Triangles on the same base, and between the same parallels, are equal.

Let the triangles ABC, DBC be on the same base BC, and between the same parallels AD, BC:
then shall the triangle ABC *be equal to the triangle* DBC.



Construction. Produce AD both ways to the points E, F; through B draw BE parallel to CA, and through C draw CF parallel to BD. [I. 31.

Proof. The parallelograms EBCA and DBCF are equal, because they are on the same base BC, and between the same parallels BC, EF. [I. 35.
Also the $\triangle$ ABC is half of the parallelogram EBCA, because the diagonal AB bisects it, [I. 34.
and the $\triangle$ DBC is half of the parallelogram DBCF, because the diagonal DC bisects it; [I. 34.
$\therefore$ the $\triangle$ ABC = the $\triangle$ DBC. [*Axiom* 7.
Wherefore, *triangles, etc.* [Q.E.D.

EXERCISES.

1. PQR is a straight line parallel and equal to the base BC of a triangle ABC and meets the sides in P and Q. Prove that the triangles BPQ, AQR are equal.

[Draw QS parallel to AB to meet BC in S; then QR = SC; $\therefore$ RC is parallel to QS or AB; $\therefore$ $\triangle^s$ AQR, RQC = $\triangle$ ARC = $\frac{1}{2}$BR = $\frac{1}{2}$BQ + $\frac{1}{2}$SR = $\triangle^s$ BPQ, RQC, etc.

2. Describe a triangle equal to a given quadrilateral ABCD.

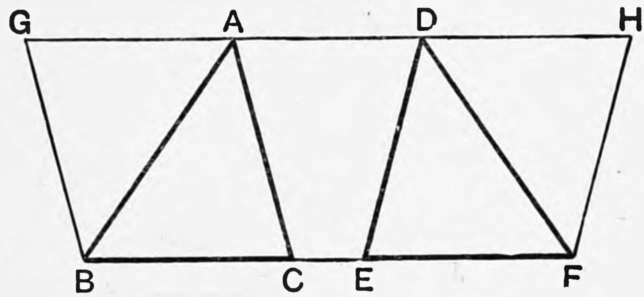
[Draw AE parallel to DB to meet BC produced in E; then AEC is the $\triangle$ required.]

42

PROPOSITION 38. THEOREM.

Triangles on equal bases, and between the same parallels, are equal to one another.

Let the triangles ABC, DEF be on equal bases BC, EF, and between the same parallels BF, AD:
then shall the triangle ABC *be equal to the triangle* DEF.



Construction. Produce AD both ways to the points G, H; through B draw BG parallel to CA, and through F draw FH parallel to ED. [I. 31.

Proof. Each of the figures GBCA, DEFH is a parallelogram, [*Definition.*
and they are equal because they are on equal bases BC, EF, and between the same parallels BF, GH. [I. 36.
Also the $\triangle$ ABC is half of the parallelogram GBCA, because the diagonal AB bisects it; [I. 34.
and the $\triangle$ DEF is half of the parallelogram DEFH, because the diagonal DF bisects it;
$\therefore$ the $\triangle$ ABC = the $\triangle$ DEF. [*Axiom* 7.

Wherefore, *triangles, etc.* [Q.E.D.

EXERCISES.

****1.** A triangle is bisected by either of its medians.

2. ABC is a triangle and E any point in the median AD; prove that the triangles ABE, ACE are equal.

****3.** If two triangles have two sides of the one equal to two sides of the other, and the contained angles supplementary, the triangles are equal in area.

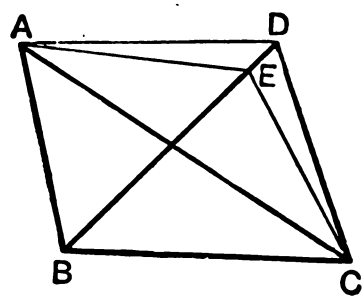
[For they can be placed so that they have one equal side of each coincident, and the other two equal sides in the same straight line. Then apply Ex. 1.]

****4.** Prove that the four triangles into which a parallelogram is divided by its diagonals are equal in area. [Use Ex. 4, page 64.]

PROPOSITION 39. THEOREM.

Equal triangles on the same base, and on the same side of it, are between the same parallels.

Let the equal triangles ABC, DBC be on the same base BC, and on the same side of it:
they shall be between the same parallels.



Construction. Join AD.
AD shall be parallel to BC.

For if it is not, let AE be parallel to BC, and let it meet BD at E, [I. 31.
and join EC.

Proof. The $\triangle$ ABC = the $\triangle$ EBC, because they are on the same base BC, and between the same parallels BC, AE. [I. 37.

But the $\triangle$ ABC = the $\triangle$ DBC; [*Hypothesis.*
$\therefore$ also the $\triangle$ DBC = the $\triangle$ EBC, [*Axiom* 1.
the greater to the less, which is impossible;
$\therefore$ AE is not parallel to BC.
In the same manner it can be shewn that no other straight line through A except AD is parallel to BC;
$\therefore$ AD is parallel to BC.

Wherefore, *equal triangles, etc.* [Q.E.D.

[For Exercises see page 72.]

PROPOSITION 40. THEOREM.

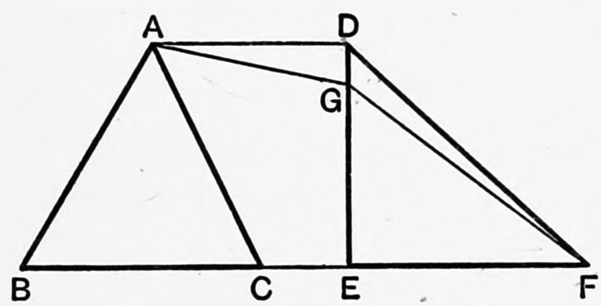
Equal triangles, on equal bases, in the same straight line, and on the same side of it, are between the same parallels.

Let the equal triangles ABC, DEF be on equal bases BC, EF, in the same straight line BF, and on the same side of it: *they shall be between the same parallels.*

Construction. Join AD.
AD shall be parallel to BF.

For if it is not, let AG be drawn parallel to BF, and let it meet ED at G, [I. 31.

and join GF.



Proof. The $\triangle$ ABC = the $\triangle$ GEF, because they are on equal bases BC, EF, and between the same parallels. [I. 38.

But the $\triangle$ ABC = the $\triangle$ DEF; [*Hypothesis.*

$\therefore$ also the $\triangle$ DEF = the $\triangle$ GEF, [*Axiom* 1.

the greater to the less, which is impossible.

$\therefore$ AG is not parallel to BF.

In the same manner it can be shewn that no other straight line through A except AD is parallel to BF;

$\therefore$ AD is parallel to BF.

Wherefore, *equal triangles, etc.* [Q.E.D.

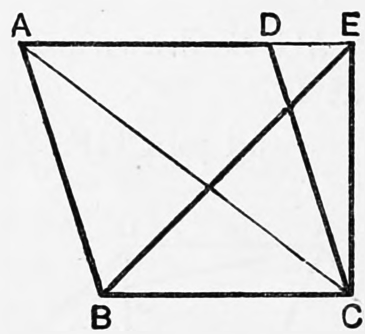
[For Exercises see page 73.]

PROPOSITION 41. THEOREM.

If a parallelogram and a triangle be on the same base and between the same parallels, the parallelogram shall be double of the triangle.

Let the parallelogram ABCD and the triangle EBC be on the same base BC, and between the same parallels BC, AE: *the parallelogram* ABCD *shall be double of the triangle* EBC.

Construction. Join AC.



Proof. The △ ABC = the △ EBC, because they are on the same base BC, and between the same parallels BC, AE. [I. 37.

But the parallelogram ABCD is double of the △ ABC, because the diagonal AC bisects the parallelogram. [I. 34.

∴ the parallelogram ABCD is also double of the △ EBC.

Wherefore, *if a parallelogram, etc.* [Q.E.D.

EXERCISES ON PROPOSITION 39.

****1.** Two straight lines AB and CD intersect at E, and the triangle AEC is equal to the triangle BED: shew that BC is parallel to AD.

****2.** The straight line which joins the middle points of two sides of any triangle is parallel to the base and is equal to half the base. [See Appendix, Art. 1.]

****3.** Straight lines joining the middle points of adjacent sides of a quadrilateral form a parallelogram. [Use Ex. 2.]

4. In the base AC of a triangle take any point D; bisect AD, DC, AB, BC at the points E, F, G, H respectively: shew that EG is equal and parallel to FH. [Use Ex. 2.]

5. Two triangles of equal area stand on the same base and on opposite sides: shew that the straight line joining their vertices is bisected by the base or the base produced. [See App., Art. 2.]

6. If a quadrilateral figure be bisected by one diagonal the second diagonal is bisected by the first. [Use Ex. 5.]

7. Any quadrilateral figure which is bisected by both of its diagonals is a parallelogram.

EXERCISES ON PROPOSITION 40.

1. A quadrilateral is divided into four triangles by its diagonals; prove that if two adjacent triangles be equal the other two triangles will also be equal.

2. The straight lines AD, BE bisecting the sides BC, AC of a triangle intersect at G: shew that AG is double of GD. [See App., Art. 7.]

EXERCISES ON PROPOSITION 41.

1. ABCD is a quadrilateral having BC parallel to AD; shew that its area is the same as that of the parallelogram which can be formed by drawing through the middle point of DC a straight line parallel to AB to meet AD and BC.

2. ABCD is a quadrilateral having BC parallel to AD, E is the middle point of DC; shew that the triangle AEB is half the quadrilateral.

3. If any point be taken within a parallelogram the sum of the triangles formed by joining the point with the extremities of a pair of opposite sides is half the parallelogram.

[Through the point draw a straight line parallel to the two opposite sides to meet the other sides.]

4. ABCD is a parallelogram; from any point P in the diagonal BD the straight lines PA, PC are drawn. Shew that the triangles PAB and PCB are equal in area.

[By Ex. 3, $\triangle$ BPC + $\triangle$ APD = $\frac{1}{2}$ABCD = $\triangle$ ABD = $\triangle$ APB + $\triangle$ APD.]

****5.** If the middle points of any two sides of a triangle be joined, the triangle so cut off is one quarter of the whole.

6. On the same side of the straight line ABC equal rectangles ABDE, ACFG are described. Prove that BG and DF are parallel.

[$\triangle$ GBF = $\frac{1}{2}$ rect. AF = $\frac{1}{2}$ rect. AD = $\triangle$ GDB; $\therefore$ etc.]

****7.** If two sides of a triangle be given the area of the triangle is greatest when the angle between them is a right angle.

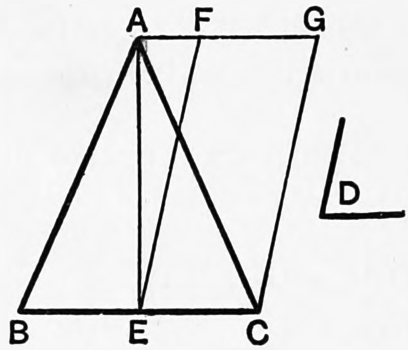
8. Through the vertices of a quadrilateral straight lines are drawn parallel to the diagonals; prove that the figure thus obtained is a parallelogram whose area is twice that of the quadrilateral.

PROPOSITION 42. PROBLEM.

To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let ABC be the given triangle, and D the given rectilineal angle:

it is required to describe a parallelogram that shall be equal to the given triangle ABC, *and have one of its angles equal to* D.



Construction. Bisect BC at E; [I. 10.
join AE, and at the point E, in the straight line EC, make the angle CEF equal to D; [I. 23.
through A draw AFG parallel to EC, and through C draw CG parallel to EF. [I. 31.
Then FECG is the parallelogram constructed as required.

Proof. The △ ABE = the △ AEC, because they are on equal bases BE, EC, and between the same parallels BC, AG. [I. 38.
∴ the △ ABC is double of the △ AEC.
But the parallelogram FECG is also double of the △ AEC, because they are on the same base EC, and between the same parallels EC, AG. [I. 41.
∴ the parallelogram FECG = the △ ABC, [*Axiom* 6.
and its angle CEF is equal to the given angle D. [*Construction.*

Wherefore *a parallelogram* FECG *has been described equal to the given triangle* ABC, *and having one of its angles* CEF *equal to the given angle* D. [Q.E.F.

EXERCISE.

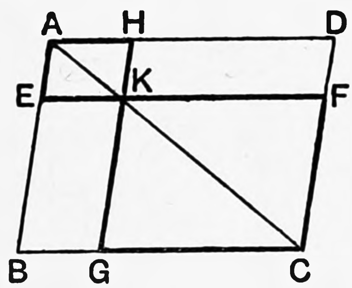
Construct a triangle equal to a given parallelogram, and having an angle equal to a given angle.

PROPOSITION 43. THEOREM.

The complements of the parallelograms which are about the diagonal of any parallelogram are equal.

Let ABCD be a parallelogram, of which the diagonal is AC; and EH, GF parallelograms about AC, that is, through which AC passes; and BK, KD the other parallelograms which make up the whole figure ABCD, and which are therefore called the complements:

the complement BK *shall be equal to the complement* KD.



Proof. Because AEKH is a parallelogram, and AK its diagonal, the △AEK = the △AHK. [I. 34.

For the same reason the △KGC = the △KFC;

∴ the two triangles AEK, KGC together = the two triangles AHK, KFC. [*Axiom* 2.

But the whole △ABC = the whole △ADC, because the diagonal AC bisects the parallelogram ABCD; [I. 34.

∴ the remainder, the complement BK, is equal to the remainder, the complement KD. [*Axiom* 3.

Wherefore, *the complements, etc.* [Q. E. D.

EXERCISES.

1. Each of the parallelograms about the diagonal of a rhombus is a rhombus.

In the figure of I. 43 prove that

2. EH, DB, GF are parallel.

[△BED = △BCE = $\frac{1}{2}$BF = $\frac{1}{2}$CH = △HCD = △BHD, etc.]

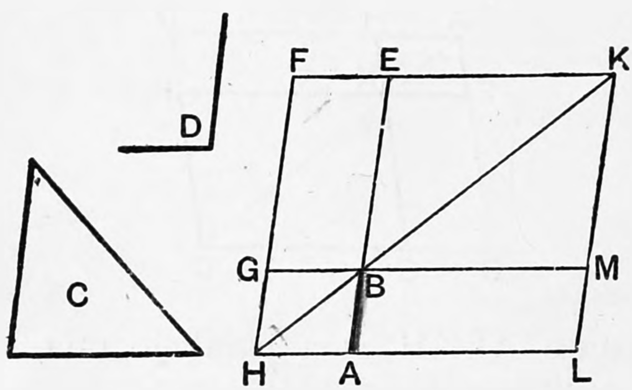
3. The triangle EKC is half of the complement BK.

4. The triangles EKC, HKC are equal.

PROPOSITION 44. PROBLEM.

To a given straight line to apply a parallelogram, which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let AB be the given straight line, C the given triangle, and D the given rectilineal angle:
it is required to apply to AB *a parallelogram equal to the triangle* C, *and having an angle equal to* D.



Construction. Make the parallelogram BEFG equal to the △ C, and having the angle EBG equal to the angle D, so that BE may be in the same straight line with AB; [I. 42.
produce FG to H;
through A draw AH parallel to BG or EF, [I. 31.
and join HB.

Because the straight line HF meets the parallels AH, EF, the angles AHF, HFE are together equal to two right angles; [I. 29.
Therefore the angles BHF, HFE are together less than two right angles;
∴ HB and FE will meet towards B and E, if produced far enough; [*Axiom* 12.
let them meet at K.
Through K draw KL parallel to EA or FH; [I. 31.
and produce HA, GB to meet KL in the points L, M.

Proof. HLKF is a parallelogram, of which the diagonal is HK; and AG, ME are parallelograms about HK; and LB, BF are the complements;

$\therefore$ LB = BF; [I. 43.

But BF = the $\triangle$ C; [*Construction.*

$\therefore$ LB = the $\triangle$ C. [*Axiom* 1.

And because the angle GBE = the angle ABM, [I. 15.

and likewise = the angle D; [*Construction.*

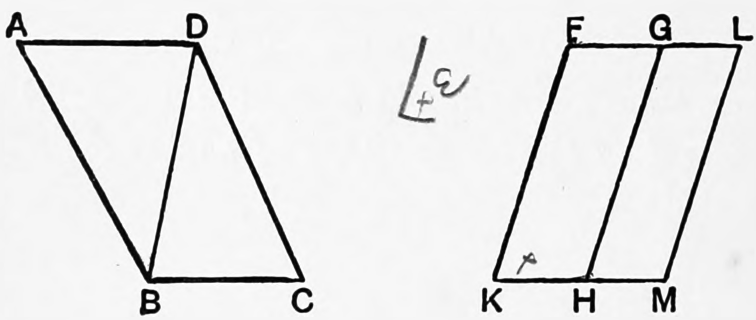
$\therefore$ the angle ABM = the angle D. [*Axiom* 1.

Wherefore *to the given straight line* AB *the parallelogram* LB *is applied, equal to the triangle* C, *and having the angle* ABM *equal to the angle* D. [Q.E.F.

PROPOSITION 45. PROBLEM.

To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.

Let ABCD be the given rectilineal figure, and E the given rectilineal angle:
it is required to describe a parallelogram equal to ABCD, *and having an angle equal to* E.



Construction. Join DB, and describe the parallelogram FH equal to the △ADB, and having the angle FKH equal to the angle E; [I. 42.
and to the straight line GH apply the parallelogram GM equal to the △DBC, and having the angle GHM equal to the angle E. [I. 44.
The figure FKML shall be the parallelogram required.

Proof. Because the angle E = each of the angles FKH, GHM, *Construction.*
the angle FKH is equal to the angle GHM. [*Axiom* 1.
Add to each of these equals the angle KHG;
∴ the angles FKH, KHG are equal to the angles KHG, GHM, [*Axiom* 2.
But FKH, KHG together = two right angles; [I. 29.
∴ KHG, GHM together = two right angles.
∴ KH is the same straight line with HM. [I. 14.

And because HG meets the parallels KM, FG, the alternate angles MHG, HGF are equal. [I. 29.
Add to each of these the angle HGL;

∴ the angles MHG, HGL together = the angles HGF, HGL. [*Axiom* 2.

But MHG, HGL together = two right angles; [I. 29.

∴ HGF, HGL together = two right angles;

∴ FG, GL are in the same straight line. [I. 14.

And because KF is parallel to HG, and HG to ML, [*Constr.*

∴ KF is parallel to ML, [I. 30.

and KM, FL are parallels; [*Construction.*

∴ KFLM is a parallelogram. [*Definition.*

And because the △ ABD = the parallelogram HF, [*Constr.*

and the △ DBC = the parallelogram GM, [*Constr.*

∴ the whole rectilineal figure ABCD = the whole parallelogram KFLM. [*Axiom* 2.

Wherefore *the parallelogram* KFLM *has been described equal to the given rectilineal figure* ABCD, *and having the angle* FKM *equal to the given angle* E. [Q.E.F.

Corollary. From this it is clear how, to a given straight line, to apply a parallelogram, which shall have an angle equal to a given rectilineal angle, and shall be equal to a given rectilineal figure, namely, by applying to the given straight line a parallelogram equal to the first triangle ABD, and having an angle equal to the given angle; and so on. [I. 44.

EXERCISE.

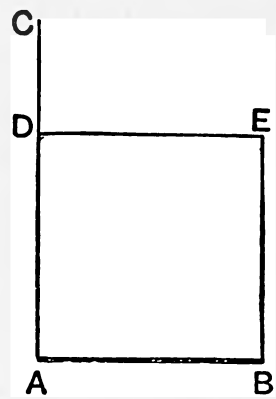
Describe a triangle equal to a given rectilineal figure.

[Let ABCDE be the given figure; join BD, AD and draw CP, EQ parallel to DB, DA respectively to meet AB produced if necessary in P and Q; then △ DCB = △ DPB and △ DEA = △ DQA [I. 37], and ∴ whole figure = △ DQP. Similarly for a figure with a larger number of sides.]

PROPOSITION 46. PROBLEM.

To describe a square on a given straight line.

Let AB be the given straight line:
it is required to describe a square on AB.



Construction. From the point A draw AC at right angles to AB, [I. 11.
and make AD equal to AB; [I. 3.
through D draw DE parallel to AB, and through B draw BE parallel to AD. [I. 31.
ADEB shall be a square.

Proof. ADEB is by construction a parallelogram;

AB = DE, and AD = BE. [I. 34.

But AB = AD. [*Construction.*

∴ BA, AD, DE, EB are all equal, and the parallelogram ADEB is therefore equilateral. [*Axiom* 1.

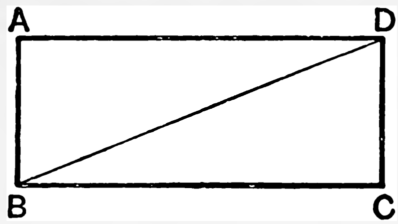
Also the angle BAD is a right angle; [*Construction.*

∴ the figure ADEB is equilateral, and it has one angle a right angle.

Therefore *it is a square,* [*Definition* 30.
and it is described on the given straight line AB. [Q.E.F.

Note to Proposition 46.

It can be proved that squares and also rectangles have all their angles right angles.



Let ABCD be a rectangle having A a right angle. Join BD. Then in the triangles ABD, CBD we have AB=CD, AD=BC, and the base BD common;

∴ the angle BCD=the angle BAD=a right angle,
and the angle ABD=the angle BDC, so that AB, CD are parallel,
and the angle ADB=the angle DBC, so that AD, BC are parallel.

Since AD, BC are parallel, the angles DAB, ABC are equal to two right angles, of which DAB is a right angle;

∴ the angle ABC is a right angle.

Similarly ADC is a right angle.

EXERCISES.

1. Prove that the sides of two equal squares must be equal.

2. The square on a given straight line is four times the square on half the line.

3. If, in the sides of AB, BC, CD, DA of a square ABCD points E, F, G, H be taken so that AE, BF, CG, DH are equal, then EFGH is a square.

4. If the diagonals of a quadrilateral are equal and bisect each other at right angles, the quadrilateral is a square.

5. On the sides AC, BC of a triangle ABC, squares ACDE, BCFH are described: shew that the straight lines AF and BD are equal.

6. Squares are described on the three sides of any triangle, the squares being all outside the triangle, and adjacent corners of the squares are joined; shew that the triangles so formed are each equal in area to the original triangle.

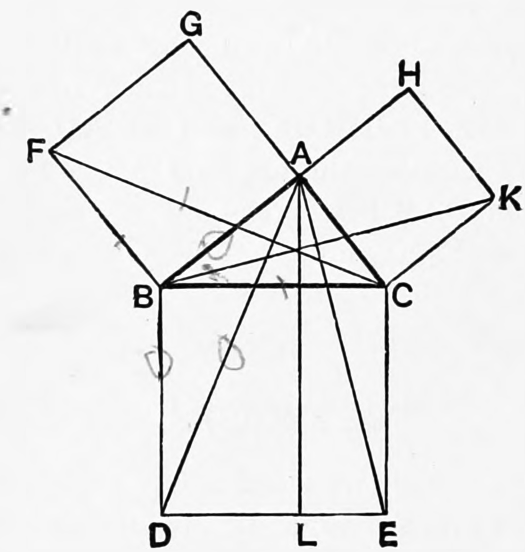
[Let ABC be the △, and BDEC, CFGA, AHKB the squares on its sides; since BCE, ACF are right ∠s, ∴ ECF, ACB are supplementary angles [I. 15, Cor. 2]; ∴ ACB, ECF are two △s with two sides equal and the included ∠s equal. Now use Ex. 3 on Prop. 38.]

PROPOSITION 47. THEOREM.

In any right-angled triangle the square which is described on the side subtending the right angle is equal to the squares described on the sides which contain the right angle.

Let ABC be a right-angled triangle, having the right angle BAC;

the square described on the side BC *shall be equal to the squares described on the sides* BA, AC.



Construction. On BC describe the square BDEC, and on BA, AC describe the squares BFGA, AHKC; [I. 46.
through A draw AL parallel to BD or CE, [I. 31.
and join AD, FC.

Proof. Because the angle BAC is a right angle, [*Hypothesis.*
and that the angle BAG is also a right angle; [*Definition* 31.
∴ CA is in the same straight line with AG. [I. 14.
Similarly, AB and AH are in the same straight line.
Now the angle DBC = the angle FBA,
for each of them is a right angle; [*Axiom* 11.
add to each the angle ABC.
∴ the whole angle DBA = the whole angle FBC. [*Axiom* 2.

Then, in the triangles DBA, FBC,

because $\begin{cases} AB = FB, & \text{[\textit{Constr.}]} \\ \text{and } BD = BC, & \text{[\textit{Constr.}]} \\ \text{and the angle } ABD = \text{the angle } FBC, & \text{[\textit{Proved.}]} \end{cases}$

$\therefore$ the $\triangle$ ABD = the $\triangle$ FBC. [I. 4.

Now the parallelogram BL is double of the $\triangle$ ABD, because they are on the same base BD, and between the same parallels BD, AL. [I. 41.

And the square GB is double of the $\triangle$ FBC, because they are on the same base FB, and between the same parallels FB, GC. [I. 41.

But the doubles of equals are equal. [*Axiom* 6.

$\therefore$ the parallelogram BL = the square GB.

Similarly, by joining AE, BK, it can be shewn that the parallelogram CL = the square CH.

$\therefore$ the whole square BDEC = the two squares GB, HC. [*Ax.* 2.

And the square BDEC is described on BC, and the squares GB, HC on BA, AC.

$\therefore$ the square described on the side BC

= the squares described on the sides BA, AC.

Wherefore, *in any right-angled triangle, etc.* [Q.E.D.

NOTE ON I. 47.

Tradition ascribed the discovery of I. 47 to Pythagoras, who flourished about 570 to 500 B.C. Many demonstrations have been given of this celebrated proposition; the following is one of the most interesting:

Let ABCD, AEFG be any two squares, placed so that their bases

may join and form one straight line. Take GH and EK each equal to AB, and join HC, CK, KF, FH.

Since GH = AB, ∴ HB = GA = FE = FG.

Since EK = AD, ∴ DK = AE = FG = HB.

∴ the △s FGH, FEK, HBC, KDC are all equal in all respects.

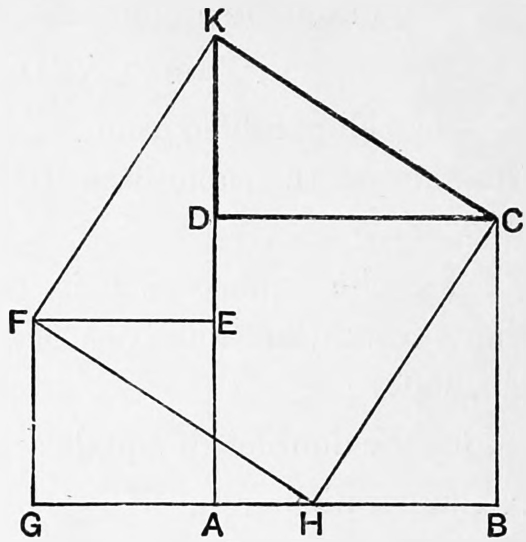
∴ the two figures AEFG, ADCB together = the fig. FHCK.

Also, since the above four △s are equal,

∴ CH, FH, FK, KC are all equal, and ∠KCD = ∠HCB and ∴ ∠HCK = ∠BCD = a rt. ∠.

∴ HCKF is a square.

Also the side CH is the hypotenuse of a right-angled triangle of which the sides CB, BH are equal to the sides of the two given squares.



This demonstration requires no proposition of Euclid after I. 32, and it shews how two given squares may be cut into pieces which will fit together so as to form a third square. *Quarterly Journal of Mathematics*, Vol. I.

It will be noted that if the △ CHB be conceived as turning round the point C as a pivot it may be rotated into the position CKD. Similarly, the △ FGH may be rotated round F into the position FED.

EXERCISES.

**1. The square described on the diagonal of a given square is twice the given square.

2. Construct a square equal to half a given square.

[Let sq. be on given str. line AB. Make ∠ABC = ∠BAC = half a rt. ∠, so that ACB = a rt. ∠ and AC = BC.

∴ sq. on AB = twice sq. on AC, etc.] [I. 48.

3. Construct a line the square on which shall be equal to the sum of three given squares.

**4. ABC is an equilateral triangle and AD is drawn perpendicular to BC; prove that the square on AD is three times the square on BD.

5. If two opposite sides of a quadrilateral be at right angles, the sum of the squares on the diagonals is equal to the sum of the squares on the other two sides.

6. The sum of the squares on the sides of a rhombus is equal to the sum of the squares on its diagonals.

7. If ABC be a triangle whose angle A is a right angle, and BE, CF be drawn bisecting the opposite sides respectively, shew that four times the sum of the squares on BE and CF is equal to five times the square on BC.

****8.** The square on the side subtending an acute angle of a triangle is less than the squares on the sides containing the acute angle.

[Let ABC be the $\triangle$, C being acute; make BCD a rt. $\angle$ and $CD=CA$. Then, by I. 24, $BA<BD$ $\therefore$ $BA^2<BD^2$, *i.e.* $<BC^2+CD^2$, *i.e.* $<BC^2+CA^2$.]

****9.** The square on the side subtending an obtuse angle of a triangle is greater than the squares on the sides containing the obtuse angle.

****10.** If the square on one side of a triangle be less than the squares on the other two sides, the angle contained by these sides is an acute angle; if greater, an obtuse angle.

In the figure of I. 47, prove that

11. AD and FC are at right angles.

[Let AD meet FC in O and BC in V. Then $\angle AOC=\angle OVC+\angle OCV=\angle BVD+\angle BDV=$ a rt. $\angle$.]

12. F, A, K are in a straight line.

13. FG, HK, and AL meet in a point.

[Let FG meet KH in U; then $\triangle^s$ AHU, CAB are equal in all respects, so that $\angle HAU=\angle ACB=$ complement of $\angle ABC=\angle BAL$, etc.]

14. BG, CH are parallel.

15. If DM, EN are drawn perpendicular to AC, AB, then AM equals AB, and AN equals AC.

[Draw DT perp^r to AB produced; then DTAM is a rectangle and $\therefore$ $AM=DT$. It can then be proved that $\triangle^s$ DTB, BAC are equal in all respects, and $\therefore$ $DT=AB$.]

16. Divide a straight line into two parts the sum of the squares on which is equal to a given square. What limit is there to the size of this given square?

[AB being given, make $\angle ABC=$ half a rt. $\angle$. With centre A and radius = side of given square describe a circle to meet BC in P. Draw PM perp^r to AB. Then AB is divided as required in M.]

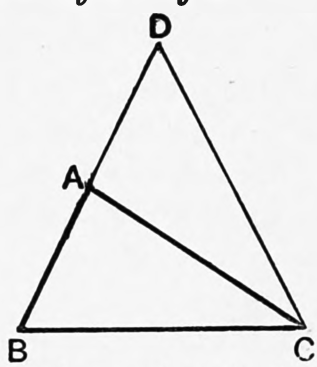
17. In a straight line AB, produced if necessary, find a point D such that the difference of the squares on AD, BD may be equal to a given square. [See App. Art. 17.]

18. Shew that a right-angled triangle can be made whose sides are proportional to the numbers 3, 4, and 5.

PROPOSITION 48. THEOREM.

If the square described on one of the sides of a triangle be equal to the squares described on the other two sides of it, the angle contained by these two sides is a right angle.

Let the square described on BC, one of the sides of the triangle ABC, be equal to the squares described on the other sides BA, AC:
the angle BAC *shall be a right angle.*



Construction. From the point A draw AD at right angles to AC; [I. 11.
make AD equal to BA, and join DC.

Proof. Because DA = BA, the square on DA = the square on BA. To each of these add the square on AC.
∴ the squares on DA, AC = the squares on BA, AC. [*Axiom* 2.
But because the angle DAC is a right angle, [*Construction.*
the square on DC = the squares on DA, AC; [I. 47.
and the square on BC = the squares on BA, AC; [*Hypothesis.*
∴ the square on DC = the square on BC; [*Axiom* 1.
∴ DC = BC.
Then, in the triangles BAC, DAC,

because { BA = AD, [*Construction.*
and AC is common,
and the base BC = the base CD;

∴ the angle DAC = the angle BAC. [I. 8.
But DAC is a right angle; [*Construction.*
∴ also BAC is a right angle. [*Axiom* 1.

Wherefore, *if the square, etc.* [Q.E.D.

BOOK II.

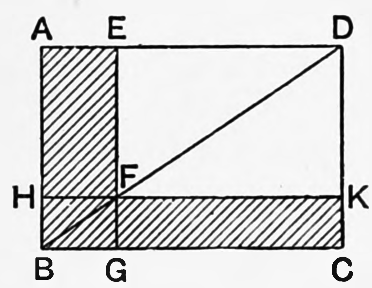
DEFINITIONS.

1. Every right-angled parallelogram, or rectangle, is said to be contained by any two of the straight lines which contain one of the right angles.

Thus the rectangle ABCD is said to be *contained* by the straight lines AB and AD, or by the straight lines BA, BC, etc.

We shall use the abbreviations "the rectangle AB, BC," or "the rect. AB, BC" for the expression "the rectangle contained by AB, BC."

2. In every parallelogram, any of the parallelograms about a diameter, together with the two complements, is called a Gnomon.



Thus the parallelogram HG, together with the complements AF, FC, is the gnomon, which is more briefly expressed by the letters AGK, or EHC, which are at the opposite angles of the parallelograms which make the gnomon.

Similarly, the figure consisting of EK together with the complements AF, FC (that is, all the figure except HG) is called the gnomon AKG.

3. When a straight line is divided into two parts, each part is called a segment by Euclid. It is found convenient to extend the meaning of the word *segment*, and to lay down the following definition: When a point P is taken in a straight line AB, or in the straight line AB produced, its distances

from the ends of the straight line are called segments of the straight line. When it is necessary to distinguish them, such segments are called **internal** or **external**, according as the point is in the straight line, or in the straight line produced.

CASE I.

CASE II.

NOTE.

There is an analogy between the first ten propositions of this book and some elementary facts in Arithmetic and Algebra.

It is shewn in Arithmetic that if one side of a rectangle contains a unit of length an exact number of times, and if an adjacent side also contains the same unit an exact number of times, the product of these units will be the number of square units in the area of the rectangle.

Thus, if the sides be respectively m inches and n inches, the area is mn square inches.

Similarly, if a square have each of its sides equal to m units, its area is m^2 square units.

We thus see that the area of a rectangle in geometry corresponds to a product of two numbers in Arithmetic or Algebra; whilst the area of a square corresponds to a square of a number.

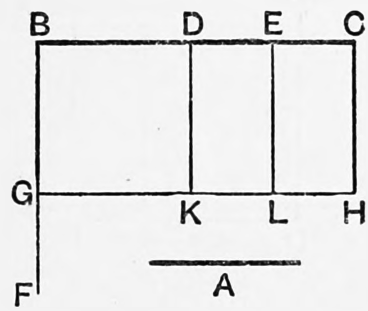
We shall add to these ten propositions the corresponding algebraic formulae. By means of the latter the student is enabled to more easily keep in his memory the results of the Propositions. The proofs in the Propositions are, however, more general than those of the formulae, since in Geometry it is not assumed that the lines spoken of are *commensurable.* We do not enter on this subject as it would lead us too far from Euclid's *Elements of Geometry* with which we are here occupied.

Owing to the above analogy the expression "the square on AB" is often abbreviated into "AB^2," and "the rectangle AB, BC" into "AB . BC." We shall sometimes use these abbreviations in the course of this book, and the student may use them in Exercises. We shall not use them in the text of the Propositions, nor should the student do so in writing out the Proposition in an Examination. The signs + and − may be used in Deductions. But the student must always carefully note that such an expression as "$AB^2 + BC . CD$" is only an *abbreviation* for "the square on AB together with the rectangle BC, CD."

Proposition 1. Theorem.

If there be two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the rectangles contained by the undivided line, and the several parts of the divided line.

Let A and BC be two straight lines; and let BC be divided into any number of parts at the points D, E:
the rectangle contained by A, BC *shall be equal to the sum of the rectangles contained by* A, BD, *by* A, DE, *and by* A, EC.



Construction. From B draw BF at right angles to BC, [I. 11.
and make BG equal to A; [I. 3.
through G draw GH parallel to BC; and through D, E, C draw DK, EL, CH, parallel to BG. [I. 31.

Proof. The rectangle BH = the sum of the rectangles BK, DL, EH.
But BH is contained by A, BC, for it is contained by GB, BC, and GB = A. [*Construction.*
And BK is contained by A, BD, for it is contained by GB, BD, and GB = A;
and DL is contained by A, DE, because DK is equal to BG, which is equal to A; [I. 34.
and in like manner EH is contained by A, EC;
∴ the rectangle contained by A, BC = the sum of the rectangles contained by A, BD, and by A, DE, and by A, EC.

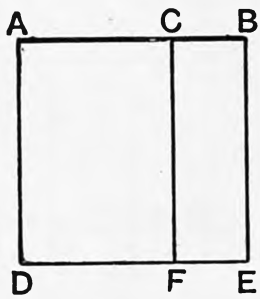
Wherefore, *if there be two straight lines, etc.* [Q.E.D.

Algebraic Formula. $a(b+c+d) = ab + ac + ad.$

PROPOSITION 2. THEOREM.

If a straight line be divided into any two parts, the rectangles contained by the whole and each of the parts are together equal to the square on the whole line.

Let the straight line AB be divided into any two parts at the point C:
the rectangle AB, BC, *together with the rectangle* AB, AC, *shall be equal to the square on* AB.



Construction On AB describe the square ADEB; [I. 46.
and through C draw CF parallel to AD or BE. [I. 31.

Proof. AE is equal to the rectangles AF, CE.
But AE is the square on AB.
Also AF is the rectangle contained by BA, AC;
for it is contained by DA, AC, of which DA = BA;
and CE is contained by AB, BC, for BE = AB;
∴ the rectangle AB, AC, together with the rectangle AB, BC, = the square on AB.

Wherefore, *if a straight line, etc.* [Q.E.D.

Algebraic Formula. Let AC be a units, and CB be b units; then

$$a(a+b)+b(a+b)=(a+b)^2$$

EXERCISE.

If a straight line be divided internally into any number of segments, the square on the straight line is equal to the sum of the rectangles contained by the straight line and the several segments.

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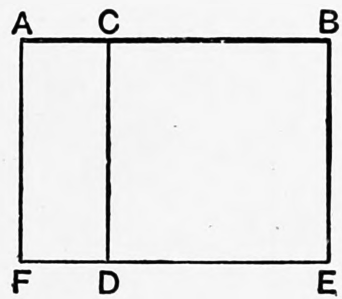
PROPOSITION 3. THEOREM.

If a straight line be divided into any two parts, the rectangle contained by the whole and one of the parts is equal to the square on that part, together with the rectangle contained by the two parts.

Let the straight line AB be divided into any two parts at the point C:
the rectangle AB, BC *shall be equal to the square on* BC, *together with the rectangle* AC, CB.



Construction. On BC describe the square CDEB; [I. 46.
produce ED to F, and through A draw AF parallel to CD or BE. [I. 31.

Proof. The rectangle AE = the rectangles AD, CE.
But AE is the rectangle contained by AB, BC; for it is contained by AB, BE, of which BE = BC;
and AD is contained by AC, CB, since CD = CB;
and CE is the square on BC;
the rectangle AB, BC = the square on BC, together with the rectangle AC, CB.

Wherefore, *if a straight line, etc.* [Q. E. D.

Algebraic Formula. Let AC be a units, and CB be b units; then

$$(a+b)b = ab + b^2.$$

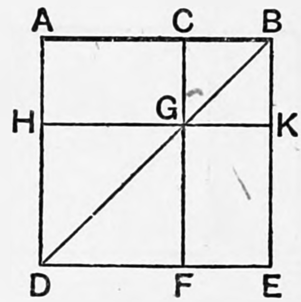
EXERCISE.

A, B, C, D are four points in a straight line taken in order; prove that the rectangle AC, BD is equal to the sum of the rectangles AB, CD and AD, BC.

PROPOSITION 4. THEOREM.

If a straight line be divided into any two parts, the square on the whole line is equal to the squares on the two parts, together with twice the rectangle contained by the two parts.

Let the straight line AB be divided into any two parts at the point C:
the square on AB *shall be equal to the squares on* AC, CB, *together with twice the rectangle* AC, CB.



Construction. On AB describe the square ADEB; [I. 46.
join BD; through C draw CGF parallel to AD or BE, and through G draw HGK parallel to AB or DE. [I. 31.

Proof. Because CF is parallel to AD, and BD meets them, the exterior angle CGB = the interior and opposite angle ADB; [I. 29.

but the angle ADB = the angle ABD, [I. 5.

because BA = AD, being sides of a square,

∴ the angle CGB = the angle CBG; [*Axiom* 1.

∴ CG = CB. [I. 6.

But CB = GK, and CG = BK; [I. 34.

∴ CK is equilateral.

Also, CBK is a right angle; [I. 46.

∴ CK is a parallelogram with all its sides equal, and one angle a right angle;

∴ it is square, and it is on the side CB. [*Definition* 31.

Similarly, HF is the square on HG which = AC. [I. 34.

∴ HF, CK are the squares on AC, CB.

Now, the complement AG = the complement GE; [I. 43.
and AG is the rectangle AC, CB, since CG = CB;
∴ GE also = the rectangle AC, CB; [*Ax.* 1.
∴ AG, GE = twice the rectangle AC, CB.

Finally, the square on AB = the figure AE,
that is, = the four figures HF, CK, AG, GE,
that is, = the squares on AC, CB, together with twice the rectangle AC, CB.

Wherefore, *if a straight line, etc.* [Q. E. D.

Corollary 1. Parallelograms about the diameter of a square are likewise squares.

Corollary 2. If, in the previous proposition, C bisect AB, the rectangle AC, CB is equal to the square on AC, so that the proposition states that, in this case, the square on AB is four times the square on AC, that is,
the square on twice a given line is four times the square on the line.

Algebraic Formula. Let AC be a units, and CB be b units; then $(a+b)^2 = a^2 + b^2 + 2ab.$

ALTERNATIVE PROOF.

Because AB is divided into two parts at C;
∴ square on AB = sum of rectangles AB, AC and AB, CB. [II. 2.
Also, by II. 3,
the rectangle AB, AC = square on AC, together with rect. AC, CB;
and the rectangle AB, CB = square on CB, together with rect. AC, CB;
∴ the square on AB = the squares on AC, CB, together with twice the rectangle AC, CB.

EXERCISES.

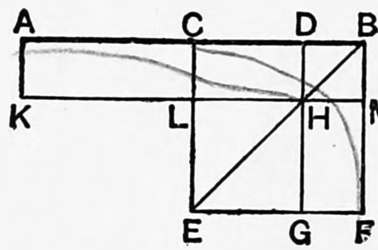
1. Prove the truth of II. 4 as follows:—On the sides AD, DE, EB of the square ADEB take points P, Q, R such that AP, DQ, and ER are each equal to BC; shew that the four triangles APC, DQP, ERQ and BCR are each one-half the rect. AC, CB, and that PQRC is the square on CP, which equals the squares on AP, AC, that is, on BC, AC.

2. In the figure of I. 1 if the circles meet again at F prove that the square on CF is three times the square on AB.

PROPOSITION 5. THEOREM.

If a straight line be divided into two equal parts and also into two unequal parts, the rectangle contained by the unequal parts, together with the square on the line between the points of section, is equal to the square on half the line.

Let the straight line AB be divided into two equal parts at the point C, and into two unequal parts at the point D: *the rectangle* AD, DB, *together with the square on* CD, *shall be equal to the square on* CB.



Construction. On CB describe the square CEFB; [I. 46. join BE; through D draw DHG parallel to CE or BF; through H draw KLM parallel to CB or EF; and through A draw AK parallel to CL or BM. [I. 31.

Proof. The complement CH = the complement HF; [I. 43.
to each of these add DM;
∴ the whole CM = the whole DF. [*Axiom* 2.
But CM = AL, because AC = CB; [I. 36.
∴ also AL = DF; to each add CH;
∴ the whole AH = DF and CH. [*Axiom* 2.

But AH is the rectangle AD, DB, for DH = DB; [II. 4, *Corollary* 1.
and DF together with CH is the gnomon CMG;
therefore the rectangle AD, DB = the gnomon CMG.
To each add the square on CD, which = LG; [II. 4, *Cor.* 1, and I. 34.
∴ the rectangle AD, DB, together with the square on CD,
= the gnomon CMG, together with LG,
that is, = the figure CEFD, which is the square on CB.

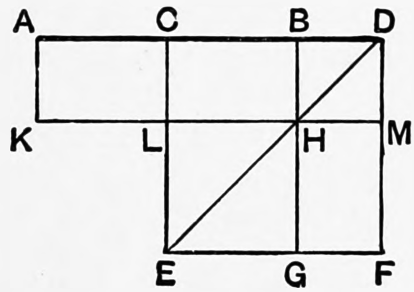
Wherefore, *if a straight line, etc.* [Q.E.D.

next

Proposition 6. Theorem.

If a straight line be bisected, and produced to any point, the rectangle contained by the whole line thus produced, and the part of it produced, together with the square on half the line bisected, is equal to the square on the straight line which is made up of the half and the part produced.

Let the straight line AB be bisected at the point C, and produced to the point D:
the rectangle AD, DB, *together with the square on* CB, *shall be equal to the square on* CD.



Construction. On CD describe the square CEFD; [I. 46.
join DE; through B draw BHG parallel to CE or DF; through H draw KLM parallel to AD or EF; and through A draw AK parallel to CL or DM. [I. 31.

Proof. Because AC = CB, [*Hypothesis.*
the rectangle AL = the rectangle CH; [I. 36.
but the complement CH = the complement HF; [I. 43.
∴ also AL = HF. [*Axiom* 1.
To each add CM;
∴ the whole AM = the gnomon CMG. [*Axiom* 2.
But AM is the rectangle contained by AD, DB, since DM = DB; [II. 4, *Corollary* 1.
∴ the rectangle AD, DB = the gnomon CMG. [*Axiom* 1.
To each add the square on CB, which = LG; [II. 4, *Cor.* 1, and I. 34.
∴ the rectangle AD, DB, together with the square on CB,
= the gnomon CMG and the figure LG,
that is, = the figure CEFD, which is the square on CD.

Wherefore, *if a straight line, etc.* [Q. E. D.

Algebraic Formula for Prop. 5. Let AC or CB be a units and CD be b units, so that

$$AD=(a+b) \text{ units and } DB=(a-b) \text{ units};$$

then $$(a+b)(a-b)+b^2=a^2.$$

Algebraic Formula for Prop. 6. Let $AC=CB=a$ units and $CD=b$ units; then

$$(a+b)(b-a)+a^2=b^2.$$

Alternative Proof of II. 5.

The sq. on CB = sqs. on CD, DB and twice the rect. CD, DB [II. 4.
= sq. on CD with rect. CD, DB and rect. DB, CB [II. 3.
= sq. on CD with rect. CD, DB and rect. AC, DB
= sq. on CD and rect. AD, DB. [II. 1.

Alternative Proof of II. 6.

The sq. on CD = sqs. on CB, BD with twice the rect. CB, BD [II. 4.
= sq. on CB with rect. CB, BD and rect. CD, BD [II. 3.
= sq. on CB with rect. AC, BD and rect. CD, BD
= sq. on CB with rect. AD, DB. [II. 1.

Note on II. 5.

From this proposition it is clear that *the difference of the squares on two unequal straight lines* AC, CD *is equal to the rectangle contained by their sum and difference.*

For the proposition states that the difference between the squares on AC, CD = the rect. AD, DB.

Also AD = the sum of AC and CD, and DB = difference between CB and CD = the diff. between AC and CD.

Again, the proposition says that *the rectangle contained by two straight lines is equal to the difference between the squares on their semi-sum and their semi-difference.*

For the rect. AD, DB = difference of sqs. on CB, CD.

Also CB = half of AB = half the sum of AD, DB,

and CD = half the difference of AD and DB,

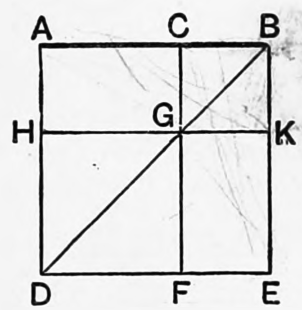
since $$AD-DB=AC+CD-DB=CB+CD-DB=2CD.$$

With a slight alteration of lettering the above is also true for the figure of II. 6.

Proposition 7. Theorem.

If a straight line be divided into any two parts, the squares on the whole line, and on one of the parts, are equal to twice the rectangle contained by the whole and that part, together with the square on the other part.

Let the straight line AB be divided into any two parts at the point C:
the squares on AB, BC *shall be equal to twice the rectangle* AB, BC, *together with the square on* AC.



Construction. On AB describe the square ADEB, and construct the figure as in the preceding propositions.

Proof. The complement AG = the complement GE; [I. 43.
to each of these add CK;
$\therefore$ the whole AK = the whole CE;
$\therefore$ AK, CE are double of AK.
But twice the rect. AB, BC is double of AK,
since BK = BC. [II. 4, *Cor.* 1.
$\therefore$ twice the rectangle AB, BC = the figures AK, CE,
that is, = gnomon AKF, together with the square CK.
To each add the square on AC, which = HF; [II. 4, *Cor.* 1 and I. 34.
$\therefore$ twice the rectangle AB, BC, together with the square on AC,
= the gnomon AKF, together with the squares CK, HF,
that is, = the whole figure ADEB, together with CK,
that is, = the squares on AB, BC.

Wherefore, *if a straight line, etc.* [Q.E.D.

Algebraic Formula. Let AB be a units and CB be b units; then $a^2+b^2=2ab+(a-b)^2$.

ALTERNATIVE PROOF.

Since AB is divided into two parts at C,
∴ sq. on AB = sqs. on AC, CB, and twice the rectangle AC, CB. [II. 4.
To each add the square on CB;
∴ the squares on AB, CB = the square on AC, together with twice the square on CB, and twice the rectangle AC, CB.

But the rect. AB, BC = sq. on CB, and the rect. AC, CB; [II. 3.
∴ twice the rect. AB, BC = twice the square on CB and twice the rect. AC, CB; [*Axiom* 6.
∴ the squares on AB, CB = the square on AC, together with twice the rectangle AB, BC.

NOTE TO PROPOSITION 7.

This proposition may be enunciated thus:

The square described on a straight line which is equal to the difference of two straight lines is less than the sum of the squares on the two straight lines by twice the rectangle contained by them.

EXERCISES.

****1.** Divide a given straight line into two parts such that the rectangle contained by them shall be the greatest possible.

[In the figure to II. 5 the rect. AD, DB is greatest when CD vanishes, *i.e.* when D is at C.]

2. The least square that can be inscribed in a given square is that which is one half the given square.

[Let ABCD be the given square, and E, F, G, H points in the sides such that AE = BF = CG = DH; then the △s AEH, EBF, FCG, GDH are all equal; ∴ the sq. EFGH is least when the △ AEH is greatest, *i.e.* when the rect. AE, AH is greatest (I. 41), *i.e.* when AE, EB is greatest, *i.e.* by Ex. 1, when E is the middle point of AB; ∴ etc.]

****3.** Of all rectangles with the same perimeter, the square has the greatest area. [Use Ex. 1.]

****4.** If ABC is an isosceles triangle and D any point on the base BC, prove that the rectangle BD, DC is equal to the difference of the squares on AC, AD.

5. The square on either of the sides about the right angle of a right-angled triangle is equal to the rectangle contained by the sum and the difference of the hypotenuse and the other side.

[Use I. 47 and the Note to II. 5.]

6. Construct a rectangle equal to the difference of two given squares. [Use II. 5 or 6.]

****7.** ABC is a triangle having a right angle at A, and AD is the perpendicular on BC; prove that

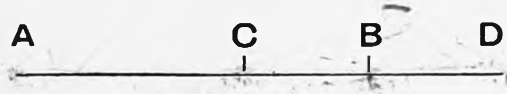
(1) AD^2 = rect. BD, DC, and (2) AC^2 = rect. BC, CD.

PROPOSITION 8. THEOREM.

If a straight line be divided into any two parts, four times the rectangle contained by the whole and one of the parts, together with the square on the other part, is equal to the square on the straight line which is made up of the whole and that part.

[Euclid's proof of this proposition is cumbrous; it is seldom read or required. The following proof is easy.]

Let the straight line AB be divided into two parts at B: *then four times the rectangle* AB, BC, *together with the square on* AC, *shall be equal to the square on the straight line made up of* AB *and* BC *together.*



Construction. Produce AB to D, making BD equal to CB.

Proof. Since AD is divided into two parts at B;
∴ the square on AD = the squares on AB, BD, together with twice the rect. AB, BD. [II. 4.
But sq. on BD = sq. on BC, and rect. AB, BD = rect. AB, BC, since BD = CB;
∴ sq. on AD = sqs. on AB, BC and twice the rect. AB, BC
But by II. 7,
the sqs. on AB, BC = sq. on AC and twice the rect. AB, BC;
∴ sq. on AD = sq. on AC, and four times the rect. AB, BC.
Also AD is made up of AB and CB together, since BD = CB.

Wherefore, *etc.*

Algebraic Formula. Let AB = a units, and CB = BD = b units; then $4ab + (a-b)^2 = (a+b)^2$.

NOTE TO PROPOSITION 8.

Since AD is the sum of the two lines AB, BC, and AC is the difference of the same two lines, this proposition may be enunciated thus:

The square on the sum of two straight lines = the square on the difference of the two lines, together with four times the rect. contained by the two lines.

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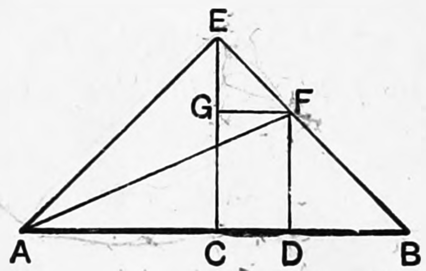
Proposition 9. Theorem.

If a straight line be divided into two equal, and also into two unequal parts, the squares on the two unequal parts are together double of the square on half the line and of the square on the line between the points of section.

Let the straight line AB be divided into two equal parts at the point C, and into two unequal parts at the point D:
the squares on AD, DB *shall be together double of the squares on* AC, CD.



Construction. From C draw CE at right angles to AB, [I. 11.
and make it equal to AC or CB, [I. 3.
and join EA, EB;
draw DF parallel to CE to meet BE in F,
and FG parallel to BA to meet CE in G, [I. 31.
and join AF.

Proof. (1) Because ACE is a right angle, [*Construction.*
the two other angles AEC, EAC are together equal to one right angle; [I. 32.
and they are equal, since CE was made equal to AC; [I. 5.
∴ each of them is half a right angle.
Similarly, each of the angles CEB, EBC is half a right angle;
∴ the whole angle AEB is a right angle.

(2) Because the angle GEF is half a right angle, and the angle EGF a right angle,
for it = the interior and opposite angle ECB; [I. 29.
∴ the remaining angle EFG is half a right angle;
∴ the angle GEF = the angle EFG, and EG = GF. [I. 6.

(3) Again, because the angle at B is half a right angle, and FDB a right angle,
for it = the interior and opposite angle ECB; [I. 29.
∴ the remaining angle BFD is half a right angle; [I. 32.
∴ the angle at B = the angle BFD, and DF = DB. [I. 6.

(4) The square on AE = the squares on AC, CE, [I. 47.
that is, = twice the square on AC, since CE = AC. [*Constr.*
Again, the square on EF = the squares on EG, GF, [I. 47.
that is, = twice the square on GF, since EG = GF; [*Proved.*
that is, = twice the square on CD, since GF = CD. [I. 34.
∴ twice the squares on AC, CD = the sqs. on AE, EF,
that is, = the square on AF, since AEF is a right angle, [I. 47.
that is, = the squares on AD, DF, since ADF is a rt. angle, [I. 47.
that is, = the sqs. on AD, DB, since DB = DF.
Wherefore, *if a straight line, etc.* [Q.E.D.

Algebraic Formula. Let AC = CB = a units and CD = b units; then

$$(a+b)^2 + (a-b)^2 = 2a^2 + 2b^2.$$

Alternative Proof of II. 9.

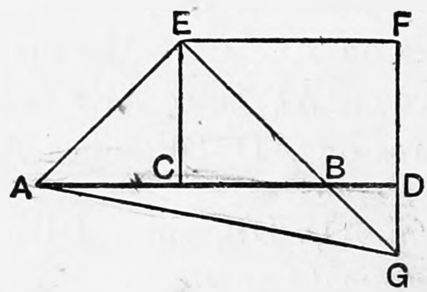
Since AD is divided at C,
∴ sq. on AC, CD + twice the rect. AC, CD = sq. on AD. [II. 4.
Again, since BC is divided internally at D,
∴ sqs. on BC, CD = twice the rect. BC, CD + sq. on BD; [II. 7.
that is, sqs. on AC, CD = twice the rect. AC, CD + sq. on BD, since AC = BC; add these results;
∴ twice the squares on AC, CD, together with twice the rect. AC, CD = the squares on AD, BD, together with twice the rect. AC, CD;
∴ twice the squares on AC, CD alone = the squares on AD, BD.

[If in the third line of this proof we read "externally" instead of "internally," it will apply also to Prop. 10.]

Proposition 10. Theorem.

If a straight line be bisected, and produced to any point, the square on the whole line thus produced, and the square on the part of it produced, are together double of the square on half the line bisected and of the square on the line made up of the half and the part produced.

Let the straight line AB be bisected at C, and produced to D: *the squares on* AD, DB *shall be together double of the squares on* AC, CD.



Construction. From C draw CE at right angles to AB, [I. 11.
and make it equal to AC or CB; [I. 3.
and join AE, EB;
draw EF parallel to AB, and DF parallel to CE. [I. 31.

Proof. (1) Because EF meets the parallels EC, FD, the angles CEF, EFD are together equal to two right angles; [I. 29.
∴ BEF, EFD are together less than two right angles;
∴ EB, FD will meet, if produced, towards B, D. [*Axiom* 12.
Let them meet at G, and join AG.

Then, because ACE is a right angle; [*Construction.*
the angles AEC, EAC together = a rt. angle, [I. 32.
and they are equal, since AC = CE; [*Construction.*
∴ each of the angles CEA, EAC is half a right angle. [I. 32.
Similarly, each of the angles CEB, EBC is half a right angle;
∴ AEB is a right angle.

(2) And because EBC is half a right angle, the vertically opposite angle DBG is also half a right angle; [I. 15.

but BDG is a right angle, because it is equal to the alternate angle DCE; [I. 29.
∴ the remaining angle DGB is half a right angle, [I. 32.
and is therefore equal to the angle DBG;
∴ also the side BD = the side DG. [I. 6.

(3) Again, because EGF is half a right angle, and the angle at F a right angle, for it = the angle ECD; [I. 34.
∴ the remaining angle FEG is half a right angle, [I. 32.
and is therefore equal to the angle EGF;
∴ also GF = FE. [I. 6.

(4) The square on AE = the squares on EC, CA, [I. 47.
that is, = twice the square on AC,
since EC = AC. [*Constr.*
Also the square on EG = the squares on GF, FE, [I. 47.
that is, = twice the square on FE,
since GF = FE, [*Proved in* (3).
that is, = twice the square on CD. [I. 34.
Hence, twice the sqs. on AC, CD = the sqs. on AE, EG,
that is, = the sq. on AG, since AEG is a rt. angle, [I. 47.
that is, = the sqs. on AD, DG, since ADG is a rt. angle, [I. 47.
that is, = the sqs. on AD, DB, since DB = DG. [*Proved in* (2).
Wherefore, *if a straight line, etc.* [Q. E. D.

Algebraic Formula. Let AC = CB = a units and CD = b units; then $(a+b)^2 + (a-b)^2 = 2a^2 + 2b^2$.

[For Exercises, see Page 105.]

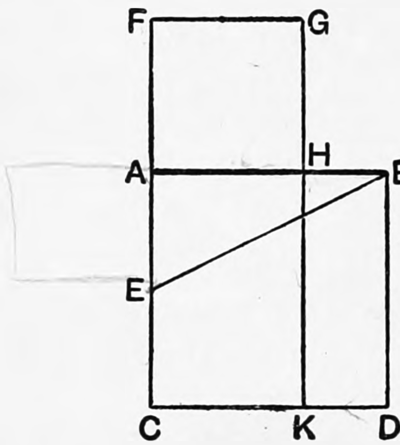
Note on Propositions 9 and 10.

These two propositions may be enunciated in one thus:
The sum of the squares on the sum and difference of two lines = twice the sum of the squares on the two lines. For in each case AD is the sum, and DB the difference of the two lines AC, CD.

PROPOSITION 11. PROBLEM.

To divide a given straight line into two parts, so that the rectangle contained by the whole and one of the parts may be equal to the square on the other part.

Let AB be the given straight line:
it is required to divide it into two parts, so that the rectangle contained by the whole and one of the parts may be equal to the square on the other part.



Construction. On AB describe the square ABDC; [I. 46.
bisect AC at E; [I. 10.
join BE; produce CA to F, and make EF equal tc EB; [I. 3.
on AF describe the square AFGH. [I. 46.
AB shall be divided at H so that the rectangle AB, BH is equal to the square on AH.

Produce GH to meet CD in K.

Proof. Because AC is bisected at E, and produced to F, the rectangle CF, FA, together with the square on AE,
= the square on EF; [II. 6.
that is, = the square on EB, since EF = EB. [*Construction.*
that is, = the sqs. on AE, AB, since EAB is a rt. angle; [I. 47.
From each of these equals take the square on AE
∴ the rect. CF, FA = the square on AB. [*Axiom* 3.
But FK is the rectangle CF, FA; for FG = FA, [*Constr.*
and AD is the square on AB;

∴ FK = AD.

Take away the common part AK,
and the remainder FH = the remainder HD. [*Axiom* 3.
But HD is the rectangle AB, BH; for AB = BD; [*Constr.*
and FH is the square on AH; [*Constr.*
∴ the rectangle AB, BH = the square on AH.

Wherefore *the straight line* AB *is divided at* H, *so that the rectangle* AB, BH *is equal to the square on* AH. [Q.E.F.

Note. When a straight line AB is divided as in the above proposition, it is said to be divided in **medial section**

EXERCISES ON PROPOSITIONS 9 and 10.

1. Divide a given straight line into two parts such that the sum of the squares on the two parts may be the least possible.

[In the figure of II. 9, $AD^2 + DB^2$ is least when CD vanishes, that is, when D is the middle point of AB.]

2. The sum of the squares on two straight lines is never less than twice the rectangle contained by them and is never less than half the square on their sum. [Use Propositions 7 and 9.]

3. If AB be bisected in C and divided unequally at D, then the sum of the squares on AD, DB is equal to twice the rectangle AD, DB together with four times the square on CD. [Use Propositions 9 and 5.]

4. C is the middle point of AB and D any point on AB produced; prove that the square on AD is equal to the square on BD, together with four times the rectangle AC, CD.

EXERCISES ON PROPOSITION 11.

****1.** *Divide a given straight line externally in medial section.*

Let AB be the given straight line. On it describe the square ACDB. Bisect AC in E and produce EC to F, making EF equal to EB. On AF, on the side remote from BD, describe the square AFGH. H shall be the point required, that is, the rectangle AB, BH shall equal the square on AH. Or completing the parallelogram CFGL the proof is similar to that of II. 11.

In the figure of II. 11 prove that

2. If CH be produced to meet BF at L, CL is at right angles to BF.

[The $\triangle^s$ HCA, FBA are equal in all respects. $\therefore$ etc.]

3. If BE and CH meet at O, AO is at right angles to CH.

[$\angle EFB = \angle EBF$. $\therefore$, by Ex. 2, $\angle ECO = \angle HOB = \angle EOC$; $\therefore EO = EC = EA$, etc.]

4. GB, FD, and AK are all parallel.

[$2\triangle GFB = FH = HD = 2\triangle DGB$; $\therefore$ etc.,

$$2\triangle GAB = 2\triangle GHB + FH = 2\triangle GHB + HD$$
$$= 2\triangle GHB + 2\triangle HKB = 2\triangle GBK;\ \therefore \text{ etc.}]$$

5. If FG meet DB in M, then CHM is a straight line.

[Use the converse of I. 43.]

6. The rectangles GD, FB, AK are all equal.

7. KF and HD are parallel.

[$2\triangle KHF = AK = DG$ [by Ex. 6] $= \triangle KDF$; $\therefore$ etc.]

8. The rectangle AH, HB is equal to the difference of the squares on AH, HB.

9. $AB^2 + BH^2 = 3AH^2$.

10. The square on the sum of AB, BH = five times the square on AH.

11. CF is divided at A in medial section.

12. The square on EF = five times the square on EA.

13. If in HA a point P be taken so that HP = HB, then AH is divided in medial section at P.

14. Shew that in a straight line, divided as in II. 11, the rectangle contained by the sum and difference of the parts is equal to the rectangle contained by the parts.

15. Produce a given straight line so that the rectangle contained by the whole straight line thus produced and the part produced may be equal to the square on the given straight line.

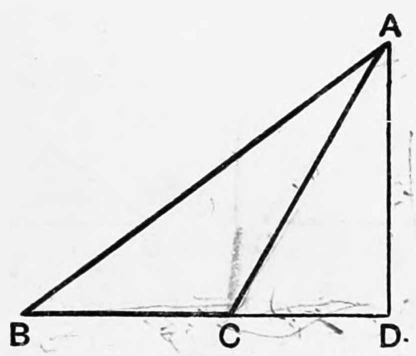
[In the figure of II. 11, if CA be the given straight line, then F is the required point; hence the required construction.]

Note. In the figures of the two following propositions the line BD is often called the **projection** of the side AB upon the base BC.

Proposition 12. Theorem.

In an obtuse-angled triangle, if a perpendicular be drawn from either of the acute angles to the opposite side produced, the square on the side subtending the obtuse angle is greater than the squares on the sides containing the obtuse angle, by twice the rectangle contained by the side on which, when produced, the perpendicular falls, and the straight line intercepted without the triangle, between the perpendicular and the obtuse angle.

Let ABC be an obtuse-angled triangle, having the obtuse angle ACB, and from the point A let AD be drawn perpendicular to BC produced:
the square on AB *shall be greater than the squares on* AC, CB, by *twice the rectangle* BC, CD.



Proof. Because BD is divided at C, the sq. on BD = sqs. on BC, CD, and twice the rect. BC, CD. [II. 4.
To each add the square on DA;
∴ the squares on BD, DA = the squares on BC, CD, DA, and twice the rectangle BC, CD. [*Axiom* 2.
But the square on BA = the squares on BD, DA, and the square on CA = the squares on CD, DA, [I. 47.
because the angle D is a right angle;
∴ the square on BA = the squares on BC, CA, and twice the rectangle BC, CD;
that is, the square on BA is greater than the squares on BC, CA by twice the rectangle BC, CD.

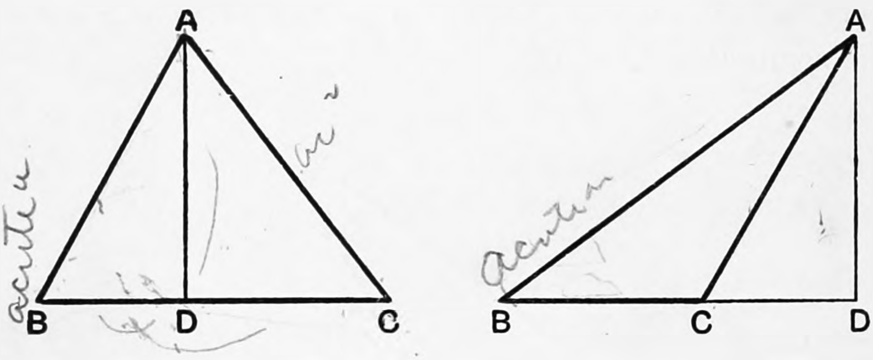
Wherefore, *in obtuse-angled triangles, etc.* [Q.E.D.

Proposition 13. Theorem.

In every triangle, the square on the side subtending an acute angle is less than the squares on the sides containing that angle, by twice the rectangle contained by either of these sides, and the straight line intercepted between the perpendicular let fall on it from the opposite angle and the acute angle.

Let ABC be any triangle, and the angle at B an acute angle; on BC let fall the perpendicular AD from the opposite angle:

the square on AC *shall be less than the squares on* CB, BA, *by twice the rectangle* CB, BD.



Proof. First, let AC be not perpendicular to BC.

In Fig. 1 BC is divided at D, and in Fig. 2 BD is divided at C; therefore, in both cases,

the squares on CB, BD = twice the rectangle CB, BD, together with the square on CD. [II. 7.

To each add the square on DA;

∴ the squares on CB, BD, DA = twice the rectangle CB, BD, together with the squares on CD, DA.

But the square on BA = the squares on BD, DA,

and the square on CA = the squares on CD, DA; [I. 47.

∴ the squares on CB, BA = twice the rectangle CB, BD, together with the square on AC.

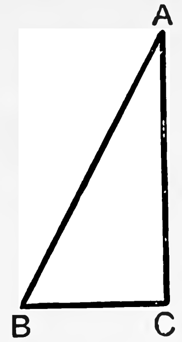
that is,

the square on AC alone is less than the squares on AB, BC by twice the rectangle CB, BD.

area of $\Delta = \frac{1}{2}$ base × alt $= \sqrt{s(s-a)(s-b)(s-c)}$

$AD = \frac{1}{2a}\sqrt{(a+b+c)(a+b-c)(a-b+c)(b+c-a)}$

Secondly, let AC be perpendicular to BC.



Then BC is the straight line between the perpendicular and the acute angle at B;
and it is clear that the squares on AB, BC
= the sq. on AC, and twice the square on BC. [I. 47 and *Ax.* 2.

Wherefore, *in every triangle, etc.* [Q.E.D.

EXERCISES ON PROPOSITIONS 12 AND 13.

****1.** *In any triangle the sum of the squares on the sides is equal to twice the square on half the base, together with twice the square on the line joining the vertex to the middle point of the base (that is, together with twice the square on the median through the vertex).*

Let D be the middle point of the base BC of the triangle ABC.

If ADB is a right angle the theorem is clear by I. 47. If not, of the two angles ADB, ADC, one is obtuse and the other acute.

Let ADB be the obtuse angle.

Draw AE perpendicular to BC.

Then by Euc. II. 12, 13,

square on AB = squares on AD, DB + twice rect. BD, DE.
and square on AC = squares on AD, DC − twice rect. DC, DE
= squares on AD, DB − twice rect. BD, DE,
since BD and DC are equal;
∴ by addition,

the squares on AB, AC = twice the squares on AD, DB.

The student will have no difficulty in drawing the figure (or see App., Art. 32). There will be two cases according as the angle ACB is acute or obtuse.

The above is a very important proposition. Many of the following exercises depend on it.

****2.** ABCD is a rectangle and P any point; prove that the squares on PA, PC are together equal to the squares on PB, PD.

**3. Four times the sum of the squares on the medians of a triangle is equal to three times the sum of the squares on the sides of the triangle.

**4. The sum of the squares on the sides of a parallelogram is equal to the sum of the squares on the diagonals.

5. The base of a triangle is given and is bisected by the centre of a given circle; if the vertex be at any point of the circumference, shew that the sum of the squares on the two sides of the triangle is invariable.

6. In any quadrilateral the squares on the diagonals are together equal to twice the sum of the squares on the straight lines joining the middle points of opposite sides. [Use Ex. 4, and Page 72, Ex. 2.]

7. If a circle be described round the point of intersection of the diameters of a parallelogram as a centre, shew that the sum of the squares on the straight lines drawn from any point in its circumference to the four angular points of the parallelogram is constant.

8. The sides of a triangle are 8, 12, and 13 inches; prove that it is acute-angled.

9. The sides of a triangle are 8, 12, and 15 inches; prove that it is obtuse-angled.

10. If the angle ACB in II. 12 become more and more obtuse, until finally A falls on BC produced, what does the proposition become?

11. The square on the base of an isosceles triangle is equal to twice the rectangle contained by either side and by the straight line intercepted between the perpendicular let fall on the side from the opposite angle and the extremity of the base.

12. If one angle of a triangle be equal to two-thirds of a right angle the square on the opposite side is less than the sum of the squares containing that angle by the rectangle contained by these two sides.

13. Find the obtuse angle of a triangle, when the square on the side opposite to the obtuse angle is greater than the sum of the squares on the sides containing it by the rectangle of the sides.

14. ABC is a triangle having the sides AB and AC equal; if AB is produced beyond the base to D so that BD is equal to AB, shew that the square on CD is equal to the square on AB, together with twice the square on BC.

15. In AB the diameter of a circle take two points C and D equally distant from the centre, and from any point E in the circumference draw EC, ED; shew that the squares on EC and ED are together equal to the squares on AC and AD. [Use Ex. 1 and II. 10.]

16. In BC the base of a triangle take D such that the squares on AB and BD are together equal to the squares on AC and CD; then the middle point of AD will be equally distant from B and C.

17. A square BDEC is described on the hypotenuse BC of a right-angled triangle ABC: shew that the squares on DA and AC are together equal to the squares on EA and AB. [Use Ex. 1.]

18. ABC is an acute-angled triangle, and BE, CF are the perpendiculars on CA, AB; prove that the rectangles AB, AF and AC, AE are equal.

[By II. 13, $AB^2 + AC^2 - 2BA \cdot AF = BC^2 = AB^2 + AC^2 - 2AC \cdot AE$; $\therefore$ etc.]

19. In a triangle ABC the angles B and C are acute: if E and F be the points where perpendiculars from the opposite angles meet the sides AC, AB, shew that the square on BC is equal to the rectangle AB, BF, together with the rectangle AC, CE.

20. Describe an isosceles obtuse-angled triangle such that the square on the largest side may be equal to three times the square on either of the equal sides. [The obtuse $\angle$ must = four-thirds of a right $\angle$.]

21. ABC is an equilateral triangle, and AB is produced to D so that BD is twice AB. Prove that the square on CD is seven times the square on AB.

22. Squares are described on the sides of any triangle; the sum of the squares on the straight lines joining adjacent corners of the squares are equal to three times the sum of the squares on the sides.

[The letters being as in I. 47, produce BC to X, so that BC = CX. The $\triangle^s$ KCE, ACX are equal in all respects, so that KE = AX;

$$\therefore KE^2 + AB^2 = AX^2 + AB^2 = 2AC^2 + 2BC^2, \text{ by Ex. 1.}$$

Similarly for HG^2, FD^2; $\therefore$ etc.]

23. The squares on the two equal sides of an isosceles triangle are together less than the squares on the two sides of any other triangle on the same base and between the same parallels. [Use Ex. 1.]

****24.** The squares on the sides of a quadrilateral are together greater than the squares on its diagonals by four times the square on the straight line joining the middle points of its diagonals.

[Let E, F be the middle points of the diagonals AC, BD of the quad[l] ABCD. Then, by Ex. 1

$$AB^2 + BC^2 = 2BE^2 + 2EC^2, \text{ and } AD^2 + DC^2 = 2DE^2 + 2EC^2;$$

$\therefore$ sum of the squares on the sides

$$= 4EC^2 + 2BE^2 + 2DE^2 = 4EC^2 + 4EF^2 + 4DF^2 = AC^2 + BD^2 + 4EF^2.]$$

25. If the squares on the sides of a quadrilateral are together equal to the squares on its diagonals it must be a parallelogram.

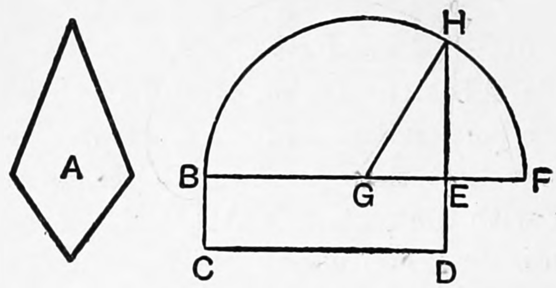
26. Construct a triangle having given its area, its base, and the sum of the squares on its sides.

[By I. 39 the vertex lies on a line parallel to the base, and by Ex. 1 its distance from the middle point of the base is given.]

PROPOSITION 14. PROBLEM.

To describe a square that shall be equal to a given rectilineal figure.

Let A be the given rectilineal figure:
it is required to describe a square that shall be equal to A.



Construction. Describe the rectangular parallelogram BCDE equal to the figure A. [I. 45.

Then, if its sides BE, ED are equal, it is a square, and what was required is now done.

But if they are not equal, produce one of them BE to F,

make EF equal to ED, [I. 3.

and bisect BF at G; [I. 10.

with centre G, and radius GB, or GF, describe the semi-circle BHF, and produce DE to meet it in H.

The square on EH shall = the figure A. Join GH.

Proof. Because BF is divided into two equal parts at G, and into two unequal parts at E,

the rectangle BE, EF, together with the square on GE

= the square on GF, [II. 5.

that is, = the square on GH, since GF = GH, [*Constr.*

that is, = the squares on GE, EH; [I. 47.

Take away the square on GE, which is common to both;

∴ the rectangle BE, EF = the square on EH. [*Axiom* 3.

But BD is the rectangle BE, EF, since EF = ED; [*Constr.*

∴ BD = the square on EH.

But BD = the rectilineal figure A; [*Construction.*

∴ the square on EH = the rectilineal figure A.

Wherefore *a square has been made equal to the given rectilineal figure* A, *namely, the square described on* EH. [Q. E. F.

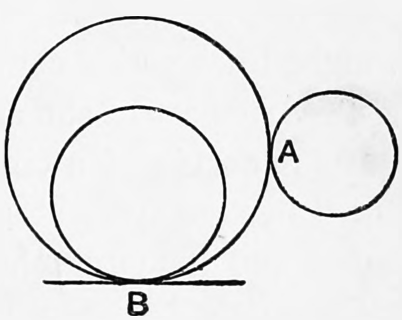
BOOK III.

DEFINITIONS.

1. Equal circles are those of which the diameters are equal, or from the centres of which the straight lines to the circumferences are equal.

[This is not a definition, but a theorem, the truth of which is evident; for, if the circles be applied to one another, so that their centres coincide, the circles must likewise coincide, since the straight lines from the centres are equal.]

2. A straight line is said to **touch** a circle when it meets the circle, and, being produced, does not cut it.



Such a straight line is called a **tangent,** and the point at which it touches the circle is called its **point of contact.**

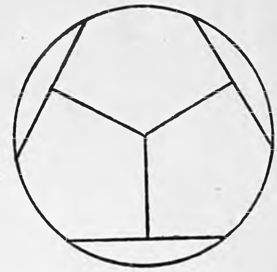
3. Circles are said to touch one another which meet but do not cut one another.

When each circle is without the other, as at A, they are said to touch **externally**; when one is within the other, as at B, they are said to touch **internally.**

4. A **secant** of a circle is a straight line which cuts the circumference in two points.

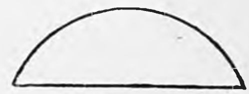
5. A **chord** of a circle is a straight line which joins any two points on the circle.

6. Chords are said to be **equally distant** from the centre of a circle when the perpendiculars drawn to them from the centre are equal.



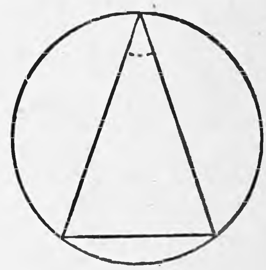
Also the chord on which the greater perpendicular falls is said to be farther from the centre.

7. A **segment** of a circle is the figure contained by a chord and the circumference it cuts off.



The chord of the segment is sometimes called its **base.**

8. An **angle in a segment** is the angle contained by two straight lines drawn from any point in the circumference of the segment to the extremities of the straight line which is the base of the segment.



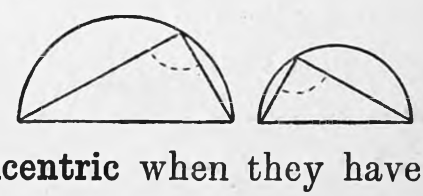
Also an angle is said to insist or stand on the circumference intercepted between the straight lines which contain the angle.

9. Any part of the circumference of a circle is called an **arc.**

10. A **sector** of a circle is the figure contained by two straight lines drawn from the centre, and the circumference between them.

The angle between the straight lines is called the angle of the sector.

11. Similar segments of circles are those in which the angles are equal, or which contain equal angles.



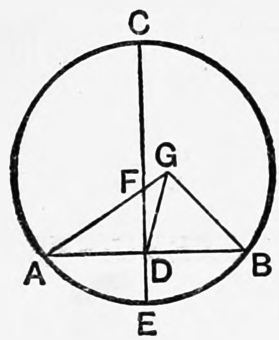
12. Two circles are said to be **concentric** when they have the same centre.

[*Note.* In the following propositions, whenever the expression "straight lines from the centre," or "drawn from the centre," occurs, it is to be understood that the lines are drawn to the circumference.]

Proposition 1. Problem.

To find the centre of a given circle.

Let ABC be the given circle:
it is required to find its centre.



Construction. Draw any chord AB, and bisect it at D; [I. 10.
from D draw DC at right angles to AB; [I. 11.
produce CD to meet the circumference at E, and bisect CE at F. [I. 10.
Then F shall be the centre of the circle ABC.

For if F be not the centre, if possible, let G be the centre; and join GA, GD, GB.

Proof. In the triangles GDA, GDB,

because $\begin{cases} \text{DA} = \text{DB}, & \text{[\textit{Construction.}} \\ \text{and DG is common to both,} \\ \text{and GA} = \text{GB,} \end{cases}$

because they are drawn from the centre G; [I. *Definition* 15.

$\therefore$ the $\angle$ADG = the $\angle$BDG. [I. 8.

these angles, being adjacent, are right angles; [I. *Def.* 10.

$\therefore$ the $\angle$BDG is a right $\angle$.

But the $\angle$BDF is also a right $\angle$; [*Construction.*

$\therefore$ the $\angle$BDG = the $\angle$BDF, [*Axiom* 11.

the less to the greater; which is impossible;

$\therefore$ G is not the centre of the circle ABC.

In the same manner it may be shewn that no other point out of the line CE is the centre;

and since CE is bisected at F, any other point in CE divides it into unequal parts, and cannot be the centre.
∴ no point but F is the centre;
that is, F is the centre of the circle ABC:
which was to be found.

Corollary. If in a circle a straight line bisect another at right angles, the centre of the circle is in the straight line which bisects the other.

EXERCISES.

****1.** The line joining the centres of two circles which meet in two points is perpendicular to the line joining the two points, and bisects it.

2. Shew that the straight lines drawn at right angles to the sides of a quadrilateral inscribed in a circle from their middle points intersect at a fixed point.

3. Find the shortest distance between two circles which do not meet.

4. Given two chords of a circle in magnitude and position, find its centre.

5. Describe a circle with a given centre cutting a given circle at the extremities of a diameter.

6. Describe a circle which shall have its centre in a given straight line and pass through two given points.

7. Describe a circle which shall pass through two given points and have a given radius.

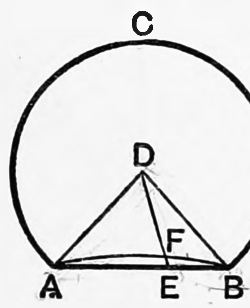
Proposition 2. Theorem.

If any two points be taken in the circumference of a circle, the straight line which joins them shall fall within the circle.

Let ABC be a circle, and A and B any two points in the circumference:

the straight line drawn from A *to* B *shall fall within the circle.*

For if it do not, let it fall, if possible, without, as AEB.



Construction. Find D, the centre of the circle ABC; [III. 1.
join DA, BD; in the arc AB take any point F, join DF, and produce it to meet AB at E.

Proof. Because DA = DB, [I. *Definition* 15.

the ∠DAB = the ∠DBA. [I. 5.

But the exterior ∠DEB of the △DAE is greater than the interior opposite ∠DAE, [I. 16.

and the ∠DAE was shewn to be equal to the ∠DBE;

∴ the ∠DEB is greater than the ∠DBE;

∴ DB is greater than DE. [I. 19.

But DB = DF; [I. *Definition* 15.

∴ DF is greater than DE, the less than the greater; which is impossible;

∴ the straight line AB does not fall without the circle.

In the same manner it may be shewn that it does not fall on the circumference.

Therefore it falls within the circle.

Wherefore, *if any two points, etc.* [Q.E.D.

PROPOSITION 3. THEOREM.

If a straight line drawn through the centre of a circle bisect a straight line in it which does not pass through the centre, it shall cut it at right angles; and if it cut it at right angles, it shall bisect it.

Let ABC be a circle; and let CD, a straight line drawn through the centre, bisect any straight line AB, which does not pass through the centre, at the point F:
CD *shall cut* AB *at right angles.*

Construction. Take E, the centre of the circle; and join EA, EB. [III. 1.

Proof. In the triangles AFE, BFE,

because { AF = FB, [*Hypothesis.*
and FE is common,
and the base EA = the base EB ; [I. *Definition* 15.

∴ the ∠AFE = the ∠BFE ; [I. 8.

∴ these angles being adjacent, are right angles. [I. *Def.* 10.

∴ CD cuts AB at right angles.

Conversely: Let CD cut AB at right angles;
CD *shall also bisect* AB, *that is,* AF *shall be equal to* FB.

The same construction being made, because EA = EB,

the ∠EAF = the ∠EBF. [I. 5.

Then in the triangles AFE, BFE,

because { the ∠EAF = the ∠EBF,
and the right ∠AFE = the right ∠BFE,
and the side EF is common ;

∴ AF = FB. [I. 26.

Wherefore, *if a straight line, etc.* [Q.E.D.

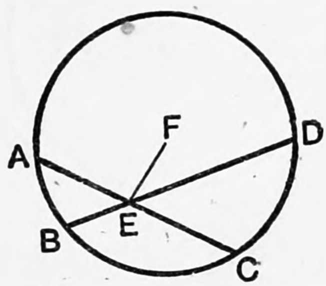
Thus

PROPOSITION 4. THEOREM.

If in a circle chords cut one another which do not both pass through the centre, they do not bisect one another.

Let ABCD be a circle, and AC, BD two chords in it, which cut one another at the point E, and do not both pass through the centre:

then AC, BD *shall not bisect one another.*



CASE I. If one of the chords pass through the centre, it is plain that it cannot be bisected by the other which does not pass through the centre.

CASE II. If neither of them pass through the centre, if possible, let AE = EC and BE = ED.

Construction. Find F, the centre of the circle, [III. 1.
and join EF.

Proof. Because FE, passing through the centre, bisects AC which does not pass through the centre; [*Hypothesis.*

∴ the ∠FEA is a right ∠. [III. 3.

Again, because FE bisects BD, [*Hypothesis.*

∴ the ∠FEB is a right ∠. [III. 3.

But the ∠FEA was shewn to be a right ∠;

∴ the ∠FEA = the ∠FEB, [*Axiom* 11.

the less to the greater; which is impossible.

∴ AC, BD do not bisect each other.

Wherefore, *if in a circle, etc.* [Q.E.D.

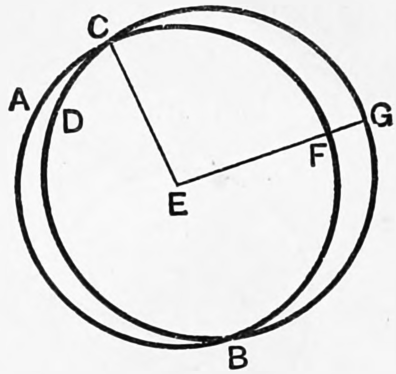
[For Exercises see next page.]

Proposition 5. Theorem.

If two circles cut one another, they shall not have the same centre.

Let the two circles ABC, CDG cut one another at the points B, C:

they shall not have the same centre.



Construction. If it be possible, let E be their centre; join EC, and draw any straight line EFG, meeting the circumferences at F and G.

Proof. Because E is the centre of the circle ABC,

$\therefore$ EC = EF. [I. *Definition* 15.

Again, because E is the centre of the circle CDG,

$\therefore$ EC = EG. [I. *Definition* 15.

$\therefore$ EF = EG, [*Axiom* 1.

the less to the greater, which is impossible;

$\therefore$ E is not the centre of the circles ABC, CDG.

Wherefore, *if two circles, etc.* [Q.E.D.

EXERCISES ON PROPOSITIONS 3 AND 4.

**1. Through a given point within a given circle draw the chord which is bisected at the point.

**2. The middle points of all parallel chords of a circle lie on a straight line which passes through the centre and is perpendicular to them.

**3. The middle points of all chords of a circle, which are of the same length, all lie on a concentric circle.

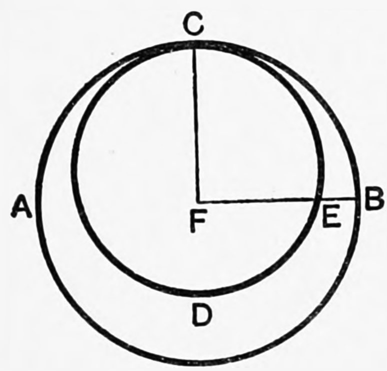
4. If two circles ABCD, ABEF cut each other in points A, B, any two parallel straight lines DAF, CBE drawn through the points of section to cut the circles are equal. [Use Ex. 2.]

Phis.

Proposition 6. Theorem.

If two circles touch one another internally, they shall not have the same centre.

Let the two circles ABC, CDE touch one another internally at the point C:
they shall not have the same centre.



Construction. If it be possible, let F be their centre; join FC, and draw any straight line FEB, meeting the circumferences at E and B.

Proof. Because F is the centre of the circle ABC,

FC = FB. [I. *Definition* 15.

Again, because F is the centre of the circle CDE,

FC = FE; [I. *Definition* 15.

∴ FE = FB,

the less to the greater, which is impossible;
∴ F is not the centre of the circles ABC, CDE.

Wherefore, *if two circles, etc.* [Q.E.D.

Note. Propositions 5 and 6 amount to this proposition: *If two circles meet in a point they cannot have the same centre;* so that circles which have the same centre and one point common must coincide altogether.

PROPOSITION 7. THEOREM.

If any point be taken in the diameter of a circle which is not the centre, of all the straight lines which can be drawn from this point to the circumference, then

(1) *the greatest is that in which the centre is, and*

(2) *the other part of the diameter is the least;*

(3) *of any others, that which is nearer to the straight line which passes through the centre is always greater than one more remote; and*

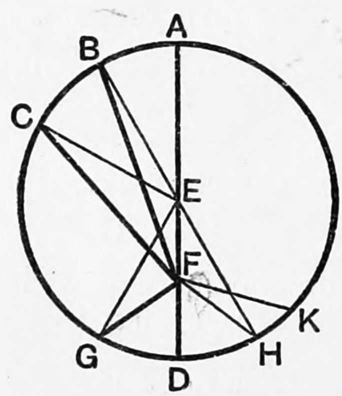
(4) *from the same point there can be drawn to the circumference two straight lines, and only two, which are equal to one another, one on each side of the shortest line.*

Let ABCD be a circle and AD its diameter, in which let any point F be taken which is not the centre; let E be the centre: of all the straight lines FB, FC, FG, etc., that can be drawn from F to the circumference,

(1) FA, *which passes through* E, *shall be the greatest, and*

(2) FD, *the other part of the diameter* AD, *shall be the least; and*

(3) *of the others* FB *shall be greater than* FC, *and* FC *than* FG.



Construction. Join BE, CE, GE.

Proof. (1) Because any two sides of a triangle are together greater than the third side, [I. 20.

therefore BE, EF are together greater than BF.

But BE = AE; [I. *Definition* 15.

∴ AE, EF are together greater than BF,

that is, AF is greater than BF.

Similarly it can be shewn that FA is greater than any other line drawn from F to the circumference;

∴ FA is the greatest of all such lines.

(2) Because GF, FE are together greater than EG, [I. 20.
and that EG = ED ; [I. *Definition* 15.

∴ GF, FE are together greater than ED.

Take away the common part FE, and the remainder GF is greater than the remainder FD.

Similarly any other straight line drawn from F can be shewn to be greater than FD ;

∴ FD is the least of all such lines.

(3) In the triangles BEF, CEF,

because { BE = CE, [I. *Definition* 15.
and EF is common,
but the ∠BEF is greater than the ∠CEF ;

∴ the base BF is greater than the base CF. [I. 24.

Similarly it may be shewn that CF is greater than GF.

(4) Also, there can be drawn two equal straight lines from the point F to the circumference, one on each side of the shortest line FD.

At the point E make the ∠FEH equal to the ∠FEG, [I. 23.
and join FH.

Then in the triangles GEF, HEF,

because { EG = EH, [I. *Definition* 15.
and EF is common,
and the ∠GEF = the ∠HEF ; [*Construction.*

∴ the base FG = the base FH. [I. 4.

But, besides FH, no other straight line can be drawn from F to the circumference equal to FG.

For, if it be possible, let FK be equal to FG.

Then, because FK = FG, and FH = FG ; [*Hypothesis.*
∴ FH = FK ; [*Axiom* 1.

that is, a line nearer to that which passes through the centre is equal to a line which is more remote,

which is impossible by what has been already shewn.

Wherefore, *if any point be taken, etc.* [Q.E.D.

PROPOSITION 8. THEOREM.

If any point be taken without a circle, and straight lines be drawn from it to the circumference, one of which passes through the centre;

(1) *of those which fall on the concave circumference, the greatest is that which passes through the centre; and of the rest, that which is nearer to the one passing through the centre is always greater than one more remote; but*

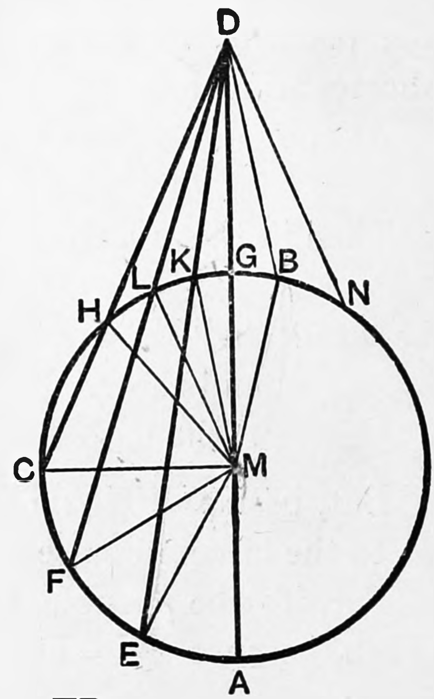
(2) *of those which fall on the convex circumference, the least is that between the point without the circle and the diameter; and of the rest, that which is nearer to the least is always less than one more remote; and*

(3) *from the same point there can be drawn to the circumference two straight lines, and only two, which are equal to one another, one on each side of the shortest line.*

Let ABC be a circle, and D any point without it, and from D let the straight lines DA, DE, DF, DC be drawn to the circumference, of which DA passes through the centre;

(1) *of those which fall on the concave circumference* AEFC, *the greatest shall be* DA, *and* DE *shall be greater than* DF, *and* DF *than* DC; *but*

(2) *of those which fall on the convex circumference* GKLH, *the least shall be* DG, *and* DK *less than* DL, *and* DL *than* DH.



Construction. Take M, the centre of the circle ABC, [III. 1.
and join ME, MF, MC, MH, ML, MK.

Proof. (1) In the △EMD the two sides EM, MD are together greater than ED. [I. 20.

But EM = AM; [I. *Definition* 15.

∴ AM, MD are together greater than ED,

that is, AD is greater than ED.

Again, in the triangles EMD, FMD,

because $\begin{cases} \text{EM} = \text{FM}, \\ \text{and MD is common,} \\ \text{but the } \angle\text{EMD is greater than the } \angle\text{FMD;} \end{cases}$

∴ the base ED is greater than the base FD. [I. 24.

Similarly, it may be shewn that FD is greater than CD. Therefore DA is the greatest, and DE greater than DF, and DF greater than DC.

(2) Again, because MK, KD are greater than MD, [I. 20.
and MK = MG, [I. *Definition* 15.
the remainder KD is greater than the remainder GD,
that is, GD is less than KD.

Also, in the △MLD, because straight lines MK, KD are drawn to a point K within the triangle from the ends of its base;
∴ MK, KD are less than ML, LD; [I. 21.
but MK = ML; [I. *Definition* 15.
∴ the remainder KD is less than the remainder LD.

Similarly, it may be shewn than LD is less than HD;
∴ DG is the least, and DK less than DL, and DL than DH.

(3) Also, there can be drawn two equal straight lines from D to the circumference, one on each side of the least line.

Construction. At the point M, in the straight line MD, make the angle DMB equal to the angle DMK, [I. 23.
and join DB.

Proof. In the triangles KMD, BMD,

because { MK = MB, [I. *Def.* 15.
and MD is common,
and the ∠DMK = the ∠DMB; [*Constr.*
∴ the base DK = the base DB. [I. 4.

But, besides DB, no other straight line can be drawn from D to the circumference equal to DK.

For, if it be possible, let DN = DK.
Then, because DB = DK, ∴ DB = DN; [*Axiom* 1.
that is, a line nearer to the least = one which is more remote, which is impossible by what has been already shewn.

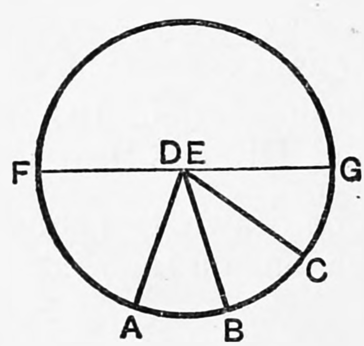
Wherefore, *if any point be taken, etc.* [Q.E.D.

Proposition 9. Theorem.

If a point be taken within a circle, from which there can be drawn more than two equal straight lines to the circumference, that point is the centre of the circle.

Let the point D be taken within the circle ABC, from which, to the circumference, there are drawn more than two equal straight lines, namely, DA, DB, DC:
the point D *shall be the centre of the circle.*

For, if not, let E be the centre.



Construction. Join DE, and produce it both ways to meet the circumference at F and G.

Proof. D is a point within the circle which is not its centre, and from it more than two equal straight lines, namely, the three straight lines DA, DB, DC, have been drawn to the circumference;

but this is impossible; [III. 7.

∴ E is not the centre of the circle ABC.

In the same manner it may be shewn that any other point than D is not the centre;
therefore D is the centre of the circle ABC.

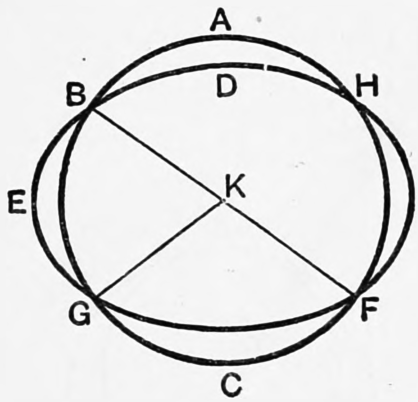
Wherefore, *if a point be taken, etc.* [Q.E.D.

Note. For other demonstrations of Propositions 9 and 10 see Notes, Pages 325 and 326.

PROPOSITION 10. THEOREM.

One circumference of a circle cannot cut another at more than two points.

If it be possible, let the circumference ABC cut the circumference DEF at more than two points, namely, at the points B, G, F.



Construction. Take K, the centre of the circle ABC, [III. 1.

and join KB, KG, KF.

Proof. Because K is the centre of the circle ABC,
∴ KB, KG, KF are all equal. [I. *Definition* 15.
And because within the circle DEF, the point K is taken, from which, to the circumference DEF, are drawn more than two equal straight lines KB, KG, KF,
∴ K is the centre of the circle DEF. [III. 9.
But K is also the centre of the circle ABC ; [*Construction.*
∴ the same point is the centre of two circles which cut one another, which is impossible. [III. 5.

Wherefore, *one circumference, etc.* [Q.E.D.

EXERCISES.

1. Two circles cannot meet in three points without coinciding entirely.

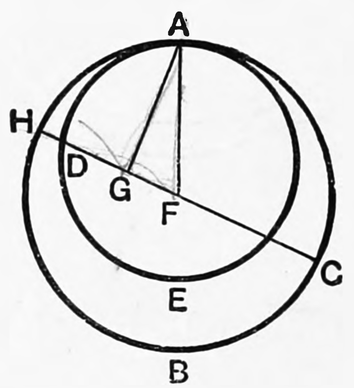
2. Through three points, which are not in the same straight line, only one circle can be drawn.

PROPOSITION 11. THEOREM.

If two circles touch one another internally, the straight line which joins their centres, being produced, shall pass through the point of contact.

Let the two circles ABC, ADE touch one another internally at the point A; let F be the centre of the circle ABC, and G the centre of the circle ADE:

the straight line FG, *being produced, shall pass through the point* A.



Proof. For, if not, let it pass otherwise, if possible, as FGDH, and join AF, AG.

Then, because AG, GF are together greater than AF, [I. 20.

and AF = HF; [I. *Definition* 15.

∴ AG, GF are together greater than HF.

Take away the common part GF;

∴ the remainder AG is greater than the remainder HG.

But AG = DG; [I. *Definition* 15.

∴ DG is greater than HG, the less than the greater,

which is impossible.

∴ FG, being produced, cannot pass otherwise than through the point A,

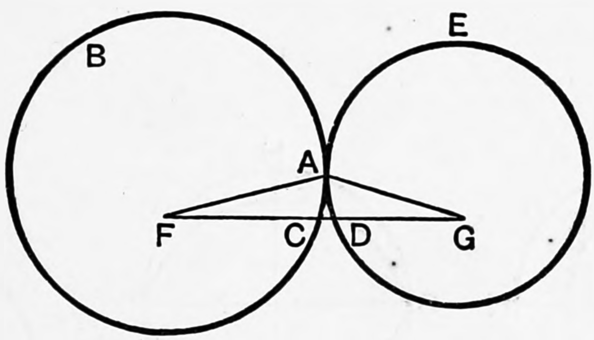
that is, it must pass through A.

Wherefore, *if two circles, etc.* [Q.E.D.

PROPOSITION 12. THEOREM.

If two circles touch one another externally, the straight line which joins their centres shall pass through the point of contact.

Let the two circles ABC, ADE touch one another externally at the point A; and let F be the centre of the circle ABC, and G the centre of the circle ADE:
the straight line FG *shall pass through the point* A.



Proof. For, if not, let it pass otherwise, if possible, as FCDG, and join FA, AG.

Then, because F is the centre of the circle ABC,

FA = FC; [I. *Definition* 15.

and because G is the centre of the circle ADE, GA = GD;
∴ FA, AG are equal to FC, DG; [*Axiom* 2.
∴ the whole FG is greater than FA, AG.
But FG is also less than FA, AG, [I. 20.

which is impossible;

∴ the straight line FG cannot pass otherwise than through the point A,

that is, it must pass through A.

Wherefore, *if two circles, etc.* [Q.E.D.

Note. Propositions 11 and 12 may be enunciated in one, thus: *If two circles touch one another, their two centres and the point of contact are in a straight line.*

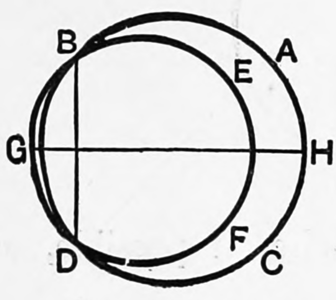
Clar

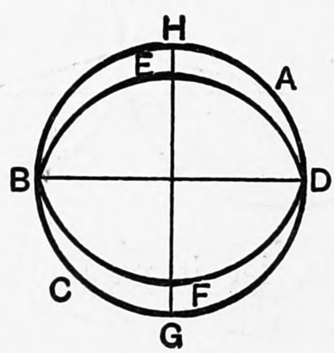
Proposition 13. Theorem.

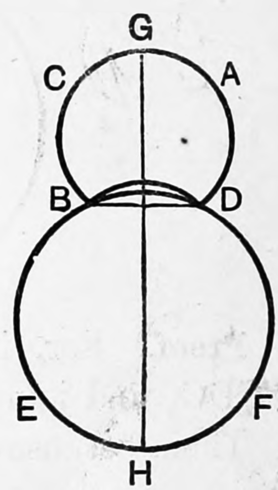
One circle cannot touch another at more points than one, whether it touches it on the inside or outside.

For, if it be possible, let the circle EBF touch the circle ABC at more points than one, namely, at B and D, internally as in Figs. 1 and 2, or externally as in Fig. 3.

Join BD, and draw GH bisecting BD at right angles. [I. 10, 11.







Proof. Because, in all three figures, BD is a chord of each circle, and GH bisects it at right angles;

∴ the centre of each circle is in GH. [III. 1, *Corollary.*

∴ GH passes through the point of contact. [III. 11, 12.

But GH does not pass through the point of contact, because the points B, D are out of the line GH,

which is absurd.

Wherefore, *one circle cannot, etc.* [Q.E.D.

EXERCISES.

****1.** If the distance between the centres of two circles be equal to the sum of their radii, the circles touch externally.

****2.** If it be equal to their difference, the circles touch internally.

3. Describe a circle with a given centre which shall touch a given circle. How many such circles can be drawn?

4. Describe a circle to pass through a given point and touch a given circle at a given point.

[Let B be the given point on the given circle whose centre is O and A the other given point. Join AB; produce OB to C, and at A make the $\angle$BAD equal to the $\angle$ABC; let AD meet OC in D; then D is the centre of the required circle, etc.]

5. Describe a circle with a given radius to pass through a given point and touch a given circle.

[The centre of the required circle is at a distance from the centre of the given circle equal to the sum of the given radius and the radius of the given circle; it is also at a distance from the given point equal to the given radius; $\therefore$ etc.]

6. Describe a circle with a given radius to touch two given circles.

7. Three circles touch one another externally at the points A, B, C; from A the straight lines AB, AC are produced to cut the circle BC at D and E. Show that DE is a diameter of BC, and is parallel to the straight line joining the centres of the other circles.

8. If in any two given circles which touch one another there be drawn two parallel diameters, an extremity of each diameter and the point of contact shall lie in the same straight line.

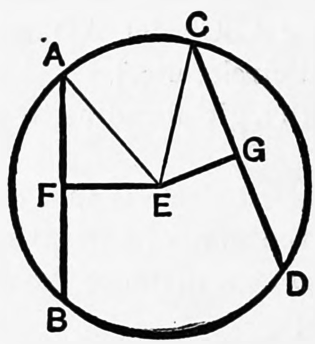
9. If circles be described on the two sides of a right-angled triangle as diameters, they will be touched by a circle whose centre is the middle point of the hypotenuse, and whose diameter is equal to the sum of the sides.

10. If a variable circle touch two fixed circles the sum, or the difference, of the distances of its centre from those of the fixed circles is equal to the sum, or the difference, of their radii.

PROPOSITION 14. THEOREM.

Equal chords in a circle are equally distant from the centre; and those which are equally distant from the centre are equal to one another.

Let the chords AB, CD in the circle ABCD be equal to one another:
they shall be equally distant from the centre.



Construction. Find E, the centre of the circle ABDC; [III. 1.
from E draw EF, EG perpendiculars to AB, CD, [I. 12.
and join EA, EC.

Proof. Because EF, passing through the centre, cuts AB, which does not pass through the centre, at right angles, it also bisects it; [III. 3.
∴ AF = FB, and AB is double of AF.
For the like reason CD is double of CG.
But AB = CD; [*Hypothesis.*
∴ AF = CG. [*Axiom* 7.
Also because AE = CE, [I. *Definition* 15.
the square on AE = the square on CE.
But the square on AE = the squares on AF, FE, because the angle AFE is a right angle; [I. 47.
similarly the square on CE = the squares on CG, GE;
∴ the squares on AF, FE = the squares on CG, GE. [*Axiom* 1.

But the square on AF = the square on CG, since AF = CG;
∴ the remaining sq. on FE = the remaining sq. on GE; [*Ax.* 3.

∴ EF = EG;

∴ AB, CD are equally distant from the centre. [III. *Def.* 6.

Conversely: Let AB, CD be chords equally distant from the centre, that is, let EF = EG;

then shall AB = CD.

For, the same construction being made, it may be shewn, as before, that AB is double of AF, and CD double of CG, and that the squares on EF, FA = the squares on EG, GC; but the square on EF = the square on EG, because EF = EG; [*Hypothesis.*

∴ the remaining sq. on FA = the remaining sq. on GC; [*Ax.* 3.

∴ AF = CG.

But AB was shewn to be double of AF, and CD double of CG;

∴ AB = CD. [*Axiom* 6.

Wherefore, *equal straight lines, etc.*

EXERCISE.

In a given circle draw a chord of given length, not greater than the diameter, so that its centre may be on a given chord.

1 lin

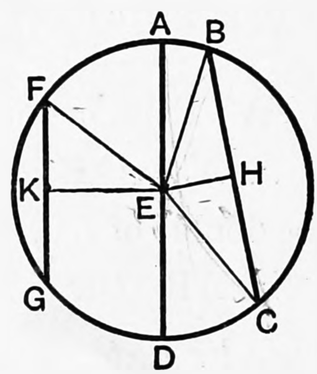
PROPOSITION 15. THEOREM.

The diameter is the greatest chord in a circle;

and, of all others, that which is nearer to the centre is always greater than one more remote;

and the greater is nearer to the centre than the less.

Let ABCD be a circle, of which AD is a diameter, and E the centre; and let BC be nearer to the centre than FG: AD *shall be greater than any chord* BC *which is not a diameter, and* BC *shall be greater than* FG.



Construction. From the centre E draw EH, EK perpendiculars to BC, FG, [I. 12.

and join EB, EC, EF.

Proof. Because AE = BE, and ED = EC, [I. *Definition* 15.

∴ AD = BE, EC; [*Axiom* 2.

but BE, EC are together greater than BC; [I. 20.

∴ also AD is greater than BC.

Also, because BC is nearer to the centre than FG, [*Hypothesis.*

EH is less than EK. [III. *Definition* 6.

Now it may be shewn, as in the preceding proposition, that BC is double of BH, and FG double of FK, and that the squares on EH, HB = the squares on EK, KF.

But the square on EH is less than the square on EK,
because EH is less than EK ;
∴ the square on HB is greater than the square on KF ;
and therefore BH is greater than FK ;
∴ BC is greater than FG.

Conversely: Let BC be greater than FG: then, the same construction being made, EH shall be less than EK.

For, because BC is greater than FG, BH is greater than FK. But the squares on BH, HE = the squares on FK, KE ;
and the square on BH is greater than the square on FK,
because BH is greater than FK ;
∴ the square on HE is less than the square on KE,
and therefore EH is less than EK.

Wherefore, *the diameter, etc.* [Q.E.D.

EXERCISE.

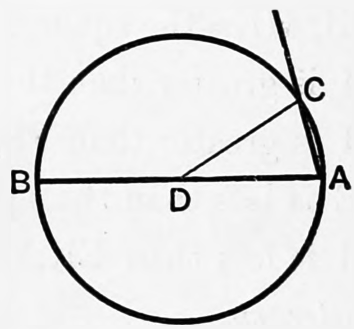
**Through a given point within a circle, draw the shortest chord.

draw it parallel to the diam.

PROPOSITION 16. THEOREM.

(1) *The straight line drawn at right angles to the diameter of a circle from the extremity of it falls without the circle; and* (2) *any other straight line drawn from the extremity must cut the circle.*

Let ABC be a circle, of which D is the centre and AB a diameter:
the straight line drawn at right angles to AB, *from its extremity* A, *shall fall without the circle.*



For, if not, let it fall, if possible, within the circle, as AC, and draw DC to the point C, where it meets the circumference.

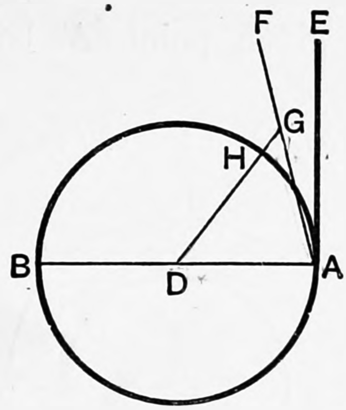
(1) **Proof.** Because DA = DC, [I. *Definition* 15.
the ∠DAC = the ∠DCA. [I. 5.
But the ∠DAC is a right ∠; [*Hypothesis.*
∴ the ∠DCA is a right ∠;
∴ the angles DAC, DCA = two right angles,
which is impossible. [I. 17.

∴ the straight line drawn from A at right angles to AB does not fall within the circle.

And in the same manner it may be shewn that it does not fall on the circumference;
∴ it must fall without the circle, as AE.

(2) If possible, let AF be another straight line drawn through A which does not cut the circle; and from the centre D draw DG perpendicular to AF; [I. 12.
let DG meet the circumference at H.

Then, because the $\angle DGA$ is a right $\angle$, [*Construction.*
the $\angle DAG$ is less than a right $\angle$; [I. 17.
$\therefore$ DA is greater than DG. [I. 19.



But $DA = DH$; [I. *Definition* 15.
$\therefore$ DH is greater than DG, the less than the greater, which is impossible;
$\therefore$ no straight line, except AE, can be drawn from the point A which does not cut the circle.

Wherefore, *the straight line, etc.* [Q.E.D.

Corollaries. (1) The straight line which is drawn at right angles to the diameter of a circle from the extremity of it touches the circle. [III. *Definition* 2.

(2) A tangent touches the circle at one point only, because if it did meet the circle at two points it would fall within it. [III. 2.

(3) There can be but one tangent at the same point.

EXERCISES.

1. Draw parallel to a given straight line a straight line to touch a given circle.

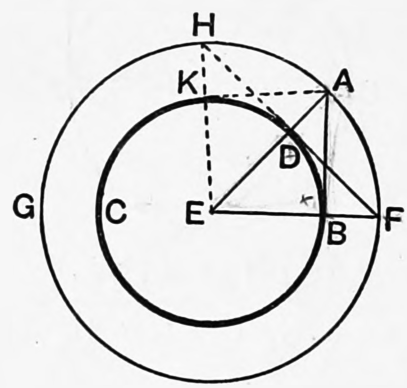
2. Draw perpendicular to a given straight line a straight line to touch a given circle.

3. Two circles have the same centre. Show that all chords of the outer circle which touch the inner circle are equal, and are bisected at the point of contact.

PROPOSITION 17. PROBLEM.

From a given point, either without or on the circumference, to draw a tangent to a given circle.

CASE I. Let the given point A be without the given circle BCD:
it is required to draw from A *a tangent to the given circle.*



Construction. Find E, the centre of the circle, [III. 1.
and join AE, cutting the given circle at D; with centre E and distance EA describe the circle AFG; from D draw HDF at right angles to EA to meet the circle AFG in F and H, [I. 11.
and join EF and EH, cutting the given circle at B and K; join AB and AK. AB and AK shall touch the circle BCD.

Proof. In the triangles AEB, FED,

because $\begin{cases} \text{AE} = \text{FE, being radii of the outer circle,} \\ \text{and EB} = \text{ED, being radii of the given circle,} \\ \text{and the angle AEB is common;} \end{cases}$

∴ the triangles are equal in all respects; [I. 4.
∴ the ∠ABE = the ∠FDE.
But the ∠FDE is a right ∠; [*Construction.*
∴ the ∠ABE is a right ∠. [*Axiom* 1.
∴ BA, being drawn at right angles to a diameter from one of its extremities B, touches the circle. [III. 16, *Corollary.*
Similarly it can be shewn that AK touches the circle at K.
Also AB and AK are drawn from the given point A. [Q.E.F.

CASE II. If the given point be in the circumference of the circle, as the point D, draw DE to the centre E, and DF at right angles to DE; then DF touches the circle. [III. 16, *Cor.*

Corollaries. (1) Two, and only two, tangents can be drawn to a circle from an external point.

(2) These two tangents are equal, subtend equal angles at the centre, and make equal angles with the line joining the centre to the given external point.

For each of the angles ABE, AKE is a right angle;
∴ the squares on AB, BE = the square on AE,
and the squares on AK, KE = the square on AE.
But the square on BE = the square on KE;
∴ the square on AB = the square on AK; ∴ AK = AB,
and the triangles AKE, ABE are equal in all respects;

∴ the $\angle$KAE = the $\angle$BAE, and
the $\angle$KEA = the $\angle$BEA.

EXERCISES.

****1.** Prove that the following construction also gives the tangents from A. *Join* EA *and bisect it in* O; *with centre* O *and radius* OA *or* OE, *describe a circle and let it meet the given circle in* P *and* Q; *join* AP *and* AQ: *these are the required tangents.*

[Join OP; since OP = OE, ∴ $\angle$OEP = $\angle$OPE; since OP = OA, ∴ $\angle$OAP = $\angle$OPA; ∴ $\angle^s$OEP, OAP = $\angle^s$OPE, OPA = $\angle$APE. Hence, by I. 32, $\angle$APE is a right angle, and AP is a tangent at P; so AQ is a tangent at Q.]

****2.** The centre of any circle which touches two given straight lines lies on the bisector of the angle between them.

****3.** A quadrilateral is described so that its sides touch a circle. Shew that two of its opposite sides are together equal to the other two sides, and conversely. [Use Cor. 2.]

4. If a hexagon, or any polygon having an even number of sides, circumscribe a circle, the sums of its alternate sides are equal.

****5.** CN is drawn from the centre C of a circle perpendicular to the chord AB. Prove that the tangents at A and B meet at a point T on CN produced.

Prove also that the rectangle CN, CT equals the square on CA.

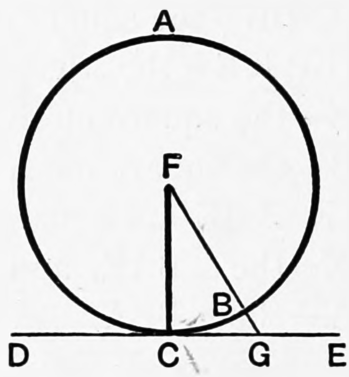
Tlie

PROPOSITION 18. THEOREM.

If a straight line touch a circle, the radius drawn from the centre of the point of contact shall be perpendicular to the line touching the circle.

Let the straight line DE touch the circle ABC at the point C;

the radius FC, *drawn from the centre* F, *shall be perpendicular to* DE.



Construction. For if not, let FG be drawn from F perpendicular to DE, meeting the circumference at B.

Proof. In the △FCG, because FGC is a right ∠, [*Hypoth.*
the ∠FCG is less than a right ∠; [I. 17.
∴ FC is greater than FG. [I. 19.
But FC = FB; [I. *Definition* 15.
∴ FB is greater than FG, the less than the greater, which is impossible.

∴ FG is not perpendicular to DE.

Similarly, it may be shewn that no other straight line from F is perpendicular to DE except FC;

∴ FC is perpendicular to DE.

Wherefore, *if a straight line, etc.* [Q.E.D.

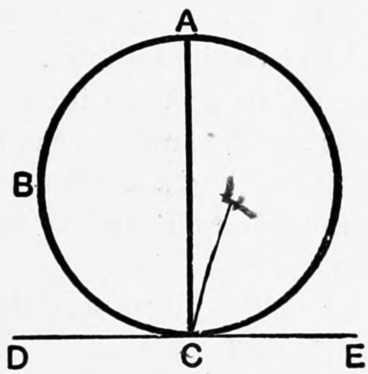
PROPOSITION 19. THEOREM.

If a straight line touch a circle, and from the point of contact a straight line be drawn at right angles to the touching line, the centre of the circle shall be in that line.

Let the straight line DE touch the circle ABC at C, and from C let CA be drawn at right angles to DE:

the centre of the circle shall be in CA.

For, if not, if possible, let F be the centre, and join CF.



Proof. Because DE touches the circle ABC, and FC is drawn from the centre to the point of contact, FC is perpendicular to DE; [III. 18.

$\therefore$ the $\angle$FCE is a right $\angle$.

But the $\angle$ACE is also a right $\angle$; [*Construction.*

$\therefore$ the $\angle$FCE = the $\angle$ACE, [*Axiom* 11.

that is, the less to the greater, which is impossible.

$\therefore$ F is not the centre of the circle ABC.

Similarly, it may be shewn that no other point out of CA is the centre; therefore the centre is in CA.

Wherefore, *if a straight line, etc.* [Q.E.D.

EXERCISES.

****1.** If two tangents to a circle be parallel, the points of contact are at the extremities of a diameter.

****2.** Shew that no parallelogram can be described about a circle except a rhombus.

[Let ABCD be a $\|^{gm}$ described about a circle, centre O, and let E, F, G, H be the points of contact of the sides AB, BC, CD, DA.

By Ex. 1, EOG, FOH are straight lines;

$\therefore$ by III. 16 Cor., OA, OC bisect the $\angle^s$ EOH, FOG, and are $\therefore$ in a straight line;

$\therefore$ $\triangle^s$ CGO, AOE are equal in all respects, so that $CG = AE = AH$;

$\therefore$ $DC = DG + GC = DH + HA = DA$, etc.]

****3.** If a quadrilateral be described about a circle, the angles subtended at the centre of the circle by any two opposite sides of the figure are together equal to two right angles.

4. Two parallel tangents to a circle intercept on a third tangent a distance which subtends a right angle at the centre.

5. The tangent at any point A of a circle meets two fixed tangents in P and Q. Prove that the points P and Q subtend a constant angle at the centre of the circle.

6. A series of circles touch a given straight line at a given point. Where will their centres all lie?

7. Describe a circle of given radius to pass through a given point and touch a given straight line.

8. A straight line is drawn touching two circles. Shew that the chords are parallel which join the points of contact and the points where the straight line through the centres meets the circumferences.

9. If two tangents TP, TQ be drawn to a circle, prove that the angle between them is double the angle between the line PQ joining their points of contact, and the diameter through either of them.

10. In the diameter of a circle produced, determine a point so that the tangent drawn from it to the circumference shall be of given length.

11. Two circles touch externally in A and the straight line BC touches them in B and C. Prove that BAC is a right angle.

12. Through a given point within or without a given circle draw a chord of given length not greater than the diameter.

[Use Page 137, Ex. 3.]

****13.** Draw a straight line to touch each of two given circles.

[See App., Art. 39.]

14. Construct a triangle, given the vertical angle, one of the sides containing it, and the perpendicular altitude.

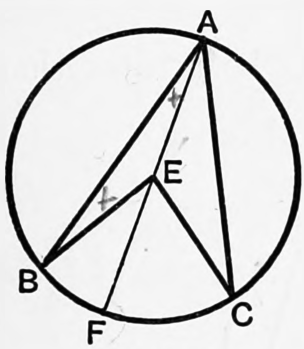
PROPOSITION 20. THEOREM.

The angle at the centre of a circle is double of the angle at the circumference on the same arc.

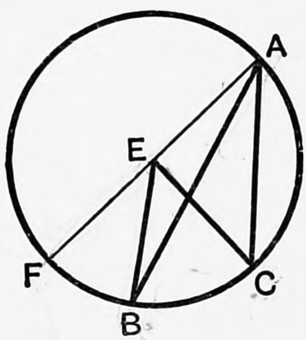
Let ABC be a circle, and BEC an angle at the centre, and BAC an angle at the circumference, which have the same arc BC for their base:

the angle BEC *shall be double of the angle* BAC.

Construction. Join AE, and produce it to F.



CASE I.



CASE II.

Proof. CASE I. Let the centre of the circle be within the angle BAC.

Then, because EA = EB, the ∠EAB = the ∠EBA; [I. 5.

∴ the angles EAB, EBA are double of the angle EAB.

But the angle BEF = the angles EAB, EBA; [I. 32.

∴ the ∠BEF is double of the ∠EAB.

Similarly, the ∠FEC is double of the ∠EAC.

∴ the whole ∠BEC is double of the whole ∠BAC.

CASE II. Let the centre E be without the ∠BAC.

As in Case I., the ∠FEC is double of the ∠FAC, and the ∠FEB, a part of the first, is double of the ∠FAB, a part of the other;

∴ the remaining ∠BEC is double of the remaining ∠BAC.

CASE III. If the centre E lie on the straight line AB it is clear, if the corresponding figure be drawn, that, as in Case I., the ∠BEC is double of the ∠BAC.

Wherefore, *the angle at the centre, etc.* [Q.E.D.

[For an important note on III. 20, see Page 326.]

PROPOSITION 21. THEOREM.

The angles in the same segment of a circle are equal to one another.

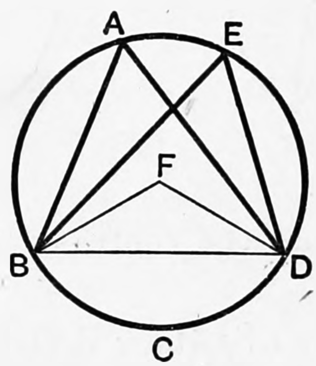
Let ABCD be a circle, and BAD, BED angles in the same segment BAED:

the angles BAD, BED *shall be equal to one another.*

Construction. Find F the centre of the circle ABCD. [III. 1.

CASE I. Let the segment BAED be greater than a semicircle.

Join BF, DF.



Proof. Because the angles BFD and BAD are angles at the centre and circumference standing on the same arc BD.

∴ the ∠BFD is double the ∠BAD. [III. 20.

Similarly the ∠BFD is double the ∠BED.

∴ the ∠BAD = the ∠BED. [*Axiom* 7.

CASE II. Let the segment BAED be not greater than a semicircle.

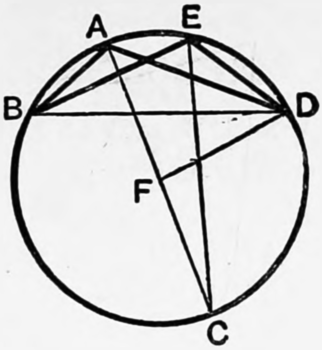
Construction. Draw AF to the centre, and produce it to meet the circumference at C, and join CE.

Proof. Since CFD and CED are angles at the centre and circumference standing on the same arc CD;

∴ the ∠CFD is double the ∠CED; [III. 20.

Similarly, the ∠CFD is double the ∠CAD;

∴ the ∠CAD = the ∠CED.



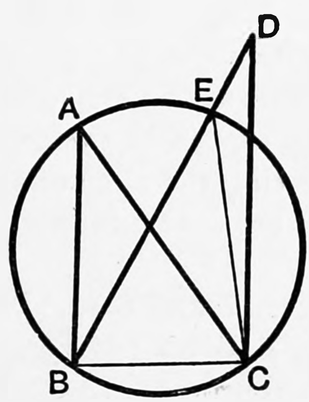
But the segment BAEC is greater than a semicircle, and therefore the ∠BAC = the ∠BEC, by Case I.

∴ the whole ∠BAD = the whole ∠BED. [*Axiom* 2.

Wherefore, *the angles in the same segment, etc.* [Q.E.D.

CONVERSE OF PROPOSITION 21.

The vertices of equal angles, standing on the same base and on the same side of it, lie on a segment of a circle passing through the extremities of the base.



Let two of the angles be ABC and DBC.

Describe a circle about ABC.

[This may be done by Prop. 5 of Book IV., which the student is now recommended to read.]

If this circle do not pass through D, let it cut BD in E, and join EC.

Then ∠BEC = ∠BAC, by the preceding proposition,

= ∠BDC, by hypothesis.

But this is impossible by I. 16,

∴ D must lie on the circle through B, A, C.

Similarly the vertex of any other such equal angle, whose arms pass through B and C, must lie on this same circle.

EXERCISES.

1. Two circles cut at A and B, and any chord PAQ is drawn terminated by the two circumferences; prove that the angle PBQ is constant.

2. If two straight lines AEB, CED in a circle intersect at E, the angles subtended by AC and BD at the centre are together double of the angle AEC.

3. Two tangents AB, AC are drawn to a circle; D is any point on the circumference outside of the triangle ABC; shew that the sum of the angles ABD and ACD is constant.

4. P, Q are any points in the circumferences of two segments described on the same straight line AB, and on the same side of it; the angles PAQ, PBQ are bisected by the straight lines AR, BR meeting at R; shew that the angle ARB is constant.

[$\angle PAR = \angle RAQ$, *i.e.* $\angle PAB - \angle RAB = \angle RAB - \angle QAB$;
$\therefore\ 2\angle RAB = \angle PAB + \angle QAB$.
So $2\angle RBA = \angle PBA + \angle QBA$;
$\therefore\ 2\angle^s RAB, RBA = \angle^s PAB, PBA, QAB, QBA$,
i.e. 4 rt. $\angle^s - 2\angle ARB = 2$ rt. $\angle^s - \angle APB + 2$ rt. $\angle^s - \angle AQB$;
$\therefore\ 2\angle ARB = \angle APB + \angle AQB =$ const.]

5. Two segments of a circle are on the same base AB, and P is any point in the circumference of one of the segments; the straight lines APD, BPC are drawn meeting the circumference of the other segment at D and C; AC and BD are drawn intersecting at Q; shew that the angle AQB is constant.

[$\angle AQB = \angle ACB - \angle CBQ$, and $\angle APB = \angle ADB + \angle CBQ$
$\therefore\ \angle^s AQB, APB = \angle^s ACB, ADB = 2\angle ACB$.
But the $\angle^s APB, ACB$ are constant; $\therefore$ etc.]

6. APB is a fixed chord passing through P a point of intersection of two circles AQP, PBR, and QPR is any other chord of the circles passing through P; shew that AQ and RB when produced meet at a constant angle.

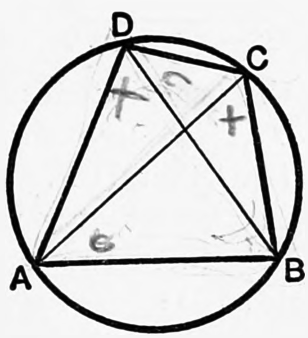
7. Two circles meet in A and B; through A two chords ACD, AC′D′ are drawn cutting the circles in C, D and C′, D′; prove that the $\triangle^s$BCD, BC′D′ are equiangular.

PROPOSITION 22. THEOREM.

The opposite angles of any quadrilateral figure inscribed in a circle are together equal to two right angles.

Let ABCD be a quadrilateral figure inscribed in the circle ABCD:

any two of its opposite angles shall be together equal to two right angles.



Construction. Join AC, BD.

Proof. The ∠CAB = the ∠CDB, because they are in the same segment CDAB; [III. 21.

and the ∠ACB = the ∠ADB, because they are in the same segment ADCB;

∴ the two angles CAB, ACB together = the whole ∠ADC. [*Axiom* 2.

To each of these equals add the ∠ABC;

∴ the three angles CAB, ACB, ABC = the two angles ABC, ADC.

But the three angles CAB, ACB, ABC, being the angles of the △ACB, are together equal to two right angles; [I. 32.

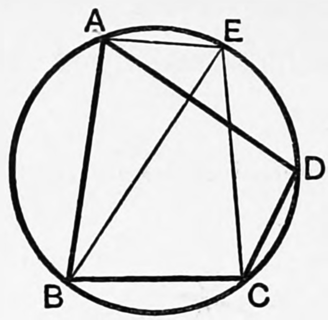
∴ also the angles ABC, ADC together = two right angles.

In the same manner it may be shewn that the angles BAD, BCD are together equal to two right angles.

Wherefore, *the opposite angles, etc.* [Q.E.D.

Converse of III. 22.

This is true and very important; namely,
If two opposite angles of a quadrilateral be together equal to two right angles, a circle may be circumscribed about the quadrilateral.



For, let ABCD denote the quadrilateral. Describe a circle round the triangle ABC, by IV. 5. Take any point E on the arc cut off by AC and on the same side of AC that D is.
Then the angles at B and E together = two right angles, by III. 22,
and the angles at B and D together = two right angles, by hypothesis;
$\therefore$ the angle at E = the angle at D;
$\therefore$ by the converse of III. 21, D is on the same segment as E.

Note. A quadrilateral, such as ABCD in the above figure, which can be inscribed in a circle is called a **Cyclic Quadrilateral**, and the four points A, B, C, D are said to be **Concyclic**.

EXERCISES.

****1.** If one side of a quadrilateral inscribed in a circle be produced, the exterior angle so formed is equal to the interior opposite angle of the quadrilateral.

2. The straight lines which bisect any angle of a quadrilateral inscribed in a circle and the opposite exterior angle meet on the circumference of the circle. [Use Ex. 1.]

3. A triangle is inscribed in a circle; shew that the sum of the angles in the three segments exterior to the triangle is equal to four right angles.

4. A quadrilateral is inscribed in a circle; shew that the sum of the angles in the four segments of the circle exterior to the quadrilateral is equal to six right angles.

5. If a polygon of an even number of sides be inscribed in a circle, the sum of the alternate angles together with two right angles is equal to as many right angles as the figure has sides.

[Let ABCDEF... be the polygon and O the centre of the circle. The $\angle ABC = 2$ rt. $\angle^s - \frac{1}{2}\angle AOC$; $\angle CDE = 2$ rt. $\angle^s - \frac{1}{2}\angle COE$, etc.;
$\therefore$ on addition, the sum of the alternate angles

$$= 2 \text{ rt. } \angle^s \times \tfrac{1}{2} \text{ no. of the sides} - \tfrac{1}{2} \text{ all the } \angle^s \text{ at } O = \text{etc.}]$$

6. Shew that the four straight lines bisecting the interior (or the exterior) angles of any quadrilateral form a quadrilateral which can be inscribed in a circle.

[Let the bisectors of the interior angles A and B, B and C, C and D, D and A of a quadrilateral meet in E, F, G, H. Then

$$\angle FEH = 2 \text{ rt. } \angle^s - \tfrac{1}{2}\angle A - \tfrac{1}{2}\angle B,$$
$$\text{and } \angle FGH = 2 \text{ rt. } \angle^s - \tfrac{1}{2}\angle C - \tfrac{1}{2}\angle D;$$

.. $\angle^s$FEH, FGH $= 4$ rt. $\angle^s$ − half the sum of the interior angles of ABCD

$$= 4 \text{ rt. } \angle^s - 2 \text{ rt. } \angle^s = 2 \text{ rt. } \angle^s; \ \therefore \text{ etc.}$$

Similarly for the bisectors of the exterior angles.]

****7.** Shew that no parallelogram except a rectangle can be inscribed in a circle.

8. D is any point on the arc BC of a circle whose centre is A; CD is produced to E; prove that the angle BDE is half the angle BAC.

9. AOB is a triangle; C and D are points in BO and AO respectively, such that the angle ODC is equal to the angle OBA; shew that a circle may be described round the quadrilateral ABCD.

10. ABCD is a quadrilateral inscribed in a circle, and the sides AB, DC when produced meet at O; shew that the triangle AOC is equiangular to the triangle BOD, and the triangle AOD to the triangle BOC.

****11.** If any two consecutive sides of a hexagon inscribed in a circle be respectively parallel to their opposite sides, the remaining sides are parallel to each other.

[Let ABCDEF be the hexagon having FA, AB parallel to CD, DE. Then $\angle FEB = 2$ rt. $\angle^s - \angle FAB$ [III. 22] $= 2$ rt. $\angle^s -$ DEC [I. 29] $= \angle EBC$.

12. ABCD is a quadrilateral inscribed in a circle; AB and CD meet in P; AD and BC meet in Q; prove that the bisectors of the angles P and Q are at right angles.

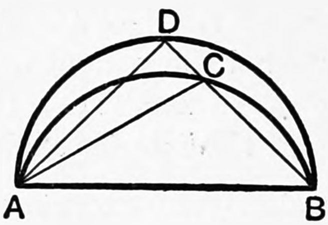
[If O be the intersection of the bisectors, and OQ meet CD in L, then

$$\angle OPC = \tfrac{1}{2}D - \tfrac{1}{2}PAD \text{ and } \angle OQD = \tfrac{1}{2}D - \tfrac{1}{2}DCQ;$$
$$\therefore \angle POQ = \angle PLQ - \angle OPC = \angle D - \angle OQD - \angle OPC$$
$$= \angle D - \tfrac{1}{2}(2\angle D - \angle PAD - \angle PCQ)$$
$$= \tfrac{1}{2}\angle PAD + \tfrac{1}{2}\angle PCQ = \text{a rt. } \angle.]$$

PROPOSITION 23. THEOREM.

On the same chord, and on the same side of it, there cannot be two similar segments of circles which do not coincide with one another.

If it be possible, on the same chord AB, and on the same side of it, let there be two similar segments of circles ACB, ADB not coinciding with one another.



Construction. Because the circle ACB cuts the circle ADB at the two points A, B, they cannot cut one another at any other point; [III. 10.

therefore one of the segments must fall within the other;

let ACB fall within ADB;

draw the straight line BCD, and join AC, AD.

Proof. Because ACB, ADB are, by hypothesis, similar segments of circles, they contain equal angles; [III. *Definition* 11.

$\therefore$ the $\angle ACB =$ the $\angle ADB$,

that is, the exterior $\angle$ of the $\triangle ACD$ is equal to the interior and opposite $\angle$, which is impossible. [I. 16.

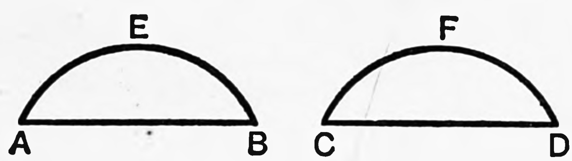
Wherefore, *on the same straight line, etc.* [Q.E.D.

PROPOSITION 24. THEOREM.

Similar segments of circles on equal chords are equal to one another.

Let AEB, CFD be similar segments of circles on the equal chords AB, CD:

the segment AEB *shall be equal to the segment* CFD.



Proof. If the segment AEB be applied to the segment CFD, so that the point A may be on the point C, and AB on CD,

then B will coincide with D, because AB = CD; [*Hyp.*

∴ AB coinciding with CD, the segment AEB must coincide with the segment CFD, [III. 23.

and is therefore equal to it.

Wherefore, *similar segments, etc.* [Q.E.D.

PROPOSITION 25. PROBLEM.

A segment of a circle being given, to describe the circle of which it is a segment.

Let ABC be the given segment of a circle:
it is required to describe the circle of which it is a segment.

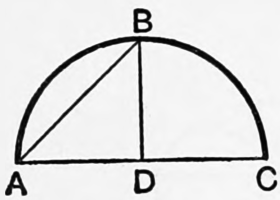


FIG. 1.

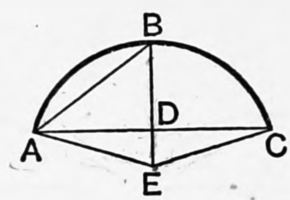


FIG. 2.

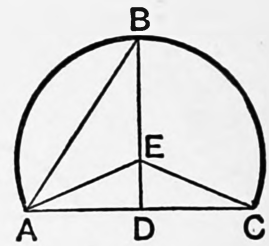


FIG. 3.

Construction. Bisect AC at D; [I. 10.
from D draw DB at right angles to AC to meet the circumference in B, [I. 11.
and join AB.

CASE I. Let the angles ABD, BAD be equal [Fig 1].

Then DB = DA; [I. 6.
but DA = DC; [*Construction.*
therefore DB = DC. [*Axiom* 1.

∴ DA, DB, DC are all equal,
and therefore D is the centre of the circle. [III. 9.

With D, and radius equal to any of the three DA, DB, DC, describe a circle; this will pass through the other points, and the circle of which ABC is a segment is described.

Also because the centre D is in AC, the segment ABC is a semicircle.

CASE II. Let the angles ABD, BAD be not equal [Figs. 2, 3].

Construction. At the point A make the angle BAE equal to the angle ABD; [I. 23.
produce BD, if necessary, to meet AE at E, and join EC.

Proof. Because the $\angle BAE =$ the $\angle ABE$, [*Construction.*

$EA = EB$. [I. 6.

Then, in the triangles ADE, CDE,

because $\begin{cases} AD = CD, & [\textit{Construction.} \\ \text{and } DE \text{ is common,} & \\ \text{and the right } \angle ADE = \text{the right } \angle CDE; & \end{cases}$

$\therefore$ the base $EA =$ the base EC. [I. 4.

But EA was shewn to be equal to EB;

therefore $EB = EC$. [*Axiom* 1.

$\therefore$ the three straight lines EA, EB, EC are all equal, and therefore a circle, whose centre is E and whose radius is any of the three EA, EB, EC, will pass through the other points, and the circle of which ABC is a segment is described. [III. 9.

And it is evident that, if the angle ABD be greater than the angle BAD, the centre E falls without the segment ABC, which is therefore less than a semicircle;

but if the angle ABD be less than the angle BAD, the centre E falls within the segment ABC, which is greater than a semicircle.

Wherefore *a segment of a circle being given, the circle has been described of which it is a segment.* [Q.E.F.

Alternative Method for Prop. 25.

The following is an alternative method: *Let* A, B, C *be three points on the arc*; *join* AB *and* BC; *bisect* AB *and* BC *in* D *and* E *respectively*; *draw* DF *perpendicular to* AB *and* EF *perpendicular to* BC, *and let them meet in* F. Then F shall be the centre of the required circle.

For by III. 1 Cor. the centre is in the straight line DF, which bisects AB at right angles.

Similarly, it is in EF.

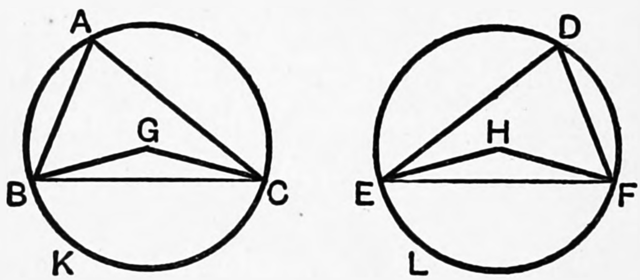
$\therefore$ it is at F, and a circle whose centre is at F and radius equal to either FA, FB, or FC will be the circle required; for as in III. 1,

$FA = FB$ and $FB = FC$.

Proposition 26. Theorem.

In equal circles the arcs which are subtended by equal angles, whether they be at the centres or circumferences, are equal.

Let ABC, DEF be equal circles; and let BGC, EHF be equal angles in them at their centres, and BAC, EDF equal angles at their circumferences:
the arc BKC *shall be equal to the arc* ELF.



Construction. Join BC, EF.

Proof. Because the circles ABC, DEF are equal, [*Hyp.*
their radii are equal; [III. *Definition* 1.

Then, in the triangles BGC, EHF,

because $\begin{cases} BG = EH, \\ \text{and } GC = HF, \\ \text{and the } \angle BGC = \text{the } \angle EHF; \end{cases}$ [*Hypothesis.*

$\therefore$ the base BC = the base EF. [I. 4.

And because the $\angle$BAC = the $\angle$EDF, [*Hypothesis.*
the segment BAC is similar to the segment EDF, [III. *Def.* 11.
and they are on equal chords BC, EF.

$\therefore$ the segment BAC = the segment EDF. [III. 24.

But the whole circle ABC = the whole circle DEF; [*Hypothesis.*

$\therefore$ the remaining segment BKC = the remaining segment ELF; [*Axiom* 3.

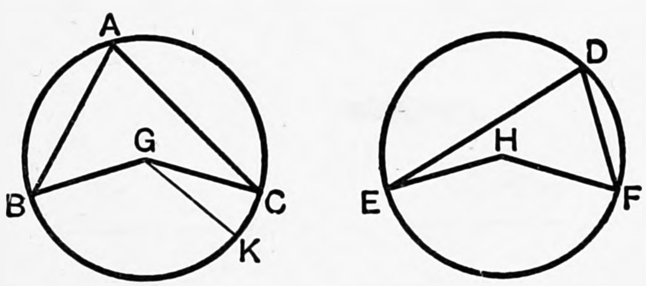
$\therefore$ the arc BKC = the arc ELF.

Wherefore, *in equal circles, etc.* [Q.E.D.

PROPOSITION 27. THEOREM.

In equal circles, the angles which stand on equal arcs are equal to one another, whether they be at the centres or circumferences.

Let ABC, DEF be equal circles, and let the angles BGC, EHF at their centres, and the angles BAC, EDF at their circumferences, stand on equal arcs BC, EF:
the angle BGC *shall be equal to the angle* EHF, *and the angle* BAC *equal to the angle* EDF.



If the ∠BGC be equal to the ∠EHF, it is manifest that the ∠BAC is also equal to the ∠EDF. [III. 20, *Axiom* 7.
But, if not, one of them must be the greater.

Let BGC be the greater, and at G make the ∠BGK equal to the ∠EHF. [I. 23.

Proof. Because the ∠BGK = the ∠EHF;
∴ the arc BK = the arc EF. [III. 26.
But the arc EF = the arc BC; [*Hypothesis*.
∴ the arc BK = the arc BC, [*Axiom* 1.
the less to the greater, which is impossible.
∴ the ∠BGC is not unequal to the ∠EHF,
that is, it is equal to it.

And the ∠ at A is half of the ∠BGC, and the ∠ at D is half of the ∠EHF; [III. 20.
∴ the ∠ at A = the ∠ at D. [*Axiom* 7.
Wherefore, *in equal circles, etc.* [Q.E.D.

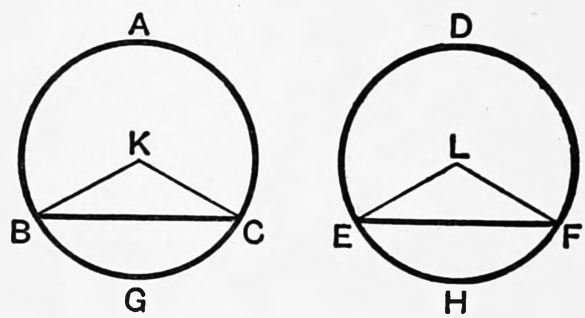
Note. Prop. 27 is the converse of Prop. 26.

PROPOSITION 28. THEOREM.

In equal circles, the arcs which are cut off by equal chords are equal, the greater equal to the greater, and the less equal to the less.

Let ABC, DEF be equal circles, and BC, EF equal chords in them, which cut off the two greater arcs BAC, EDF, and the two less arcs BGC, EHF:

the greater arc BAC *shall be equal to the greater arc* EDF, *and the less arc* BGC *equal to the less arc* EHF.



Construction. Find K, L, the centres of the circles, [III. 1.

and join BK, KC, EL, LF.

Proof. Because the circles are equal, [*Hypothesis.*

their radii are equal; [III. *Definition* 1.

Then, in the triangles BKC, ELF,

because $\begin{cases} BK = EL, \\ \text{and } KC = LF, \\ \text{and the base } BC = \text{the base } EF, \end{cases}$ [*Hypothesis.*

$\therefore$ the $\angle BKC =$ the $\angle ELF$; [I. 8.

$\therefore$ the arc BGC = the arc EHF. [III. 26.

But the circumference ABGC = the circumference DEHF; [*Hypothesis.*

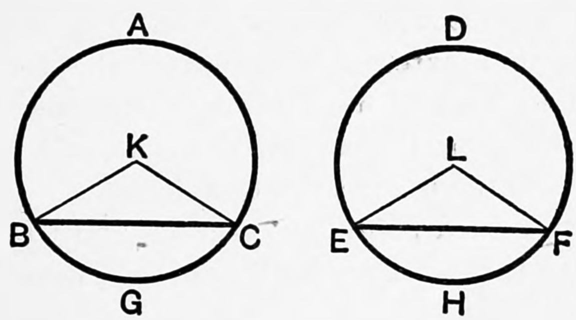
$\therefore$ the remaining arc BAC = the remaining arc EDF. [*Ax.* 3.

Wherefore, *in equal circles, etc.* [Q.E.D.

PROPOSITION 29. THEOREM.

In equal circles, the chords which cut off equal arcs are equal.

Let ABC, DEF be equal circles, and let BGC, EHF be equal arcs in them, and join BC, EF:
the chord BC *shall be equal to the chord* EF.



Construction. Find K, L, the centres of the circles, [III. 1.
and join BK, KC, EL, LF.

Proof. Because the arc BGC = the arc EHF, [*Hypothesis.*
the angle BKC = the angle ELF. [III. 27.
And because the circles ABC, DEF are equal, [*Hypothesis.*
their radii are equal; [III. *Definition* 1.
∴ the two sides BK, KC = the two sides EL, LF,
and they contain equal angles; [*Proved.*
∴ the base BC = the base EF. [I. 4.
Wherefore, *in equal circles, etc.* [Q.E.D.

Note 1. Prop. 29 is the converse of Prop. 28.

Note 2. The Propositions 26–29 tell us that in equal circles
If two angles are equal, so also are the chords and arcs subtended;
If two chords are equal, so also are the angles subtended and the arcs cut off;
If two arcs are equal, so also are the angles and chords.

Note 3. It is clear that Props. 26–29 could easily be proved by *Superposition.*

EXERCISES.

1. The straight lines joining the extremities of the chords of two equal arcs of a circle, towards the same parts, are parallel to each other.

2. The straight lines in a circle which join the extremities of two parallel chords are equal to each other.

****3.** The straight lines which bisect the vertical angles of all triangles on the same base and on the same side of it, and having equal vertical angles, all intersect at the same point.

4. The internal and external bisectors of the vertical angle of a triangle inscribed in a circle meet the circumference again in points which are equidistant from the ends of the base, and which lie on the straight line bisecting the base at right angles.

5. If A, B be two fixed points on a circle, and CD a chord of constant length ; prove that the intersections of AD, BC and of AC, BD lie on two fixed circles.

[Since CD is constant in length,

$\therefore$ the $\angle^s$CAD, CBD are constant [Props. 28, 27].

Hence if AD, BC meet in O, then

$$\angle AOB = \angle ACB + \angle DAC = \text{const.}; \quad \therefore \text{ etc.}]$$

6. AB is a common chord of two circles ; through C any point of one circumference straight lines CAD, CBE are drawn terminated by the other circumference ; shew that the arc DE is invariable.

7. If two equal circles cut each other, and if through one of the points of intersection a straight line be drawn terminated by the circles, the straight lines joining its extremities with the other point of intersection are equal.

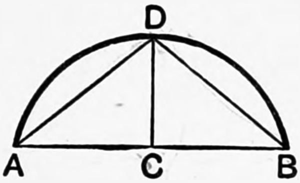
8. The straight line joining the feet of the perpendiculars from any point of a circle upon two diameters given in position is constant.

Conversely, if a straight line of given length move so that its ends slide along two given straight lines, the two perpendiculars at its ends to the given lines meet on a fixed circle.

PROPOSITION 30. PROBLEM.

To bisect a given arc, that is, to divide it into two equal parts.

Let ADB be the given arc :
it is required to bisect it.



Construction. Join AB, and bisect it at C ; [I. 10.
from the point C draw CD at right angles to AB meeting the arc at D. [I. 11.

The arc ADB shall be bisected at the point D.

Join AD, DB.

Proof. In the triangles ACD, BCD,

because $\begin{cases} AC = CB, & \text{[Construction.} \\ \text{and } CD \text{ is common to both,} & \\ \text{and the right } \angle ACD = \text{the right } \angle BCD\,; & \text{[Const.} \end{cases}$

$\therefore$ the base AD = the base BD. [I. 4.

But equal straight lines cut off equal arcs, the greater equal to the greater, and the less equal to the less, [III. 28.
and each of the arcs AD, DB is less than a semicircle, because DC, if produced, is a diameter ; [III. 1, *Cor.*

$\therefore$ the arc AD = the arc DB.

Wherefore, *the given arc is bisected at D.* [Q.E.F.

PROPOSITION 31. THEOREM.

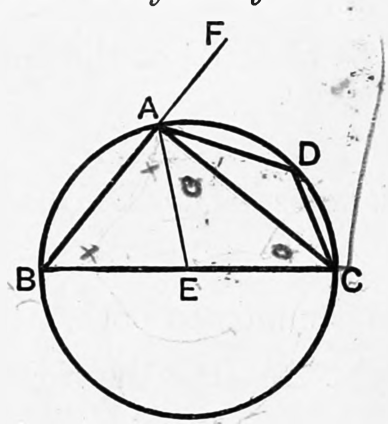
An angle in a semicircle is a right angle,
but an angle in a segment greater than a semicircle is less than a right angle,
and an angle in a segment less than a semicircle is greater than a right angle.

Let ABCD be a circle, of which BC is a diameter and E the centre; and draw CA, dividing the circle into the segments ABC, ADC:

(1) *the angle in the semicircle* BAC *shall be a right angle;*

(2) *the angle in the segment* ABC, *which is greater than a semicircle, shall be less than a right angle;*

(3) *the angle in the segment* ADC, *which is less than a semicircle, shall be greater than a right angle.*



Construction. Join AE, and produce BA to F.
Take any point D on the arc ADC, and join AD, DC.

Proof. (1) Because EA = EB, [I. *Definition* 15.
the ∠EAB = the ∠EBA; [I. 5.
and, because EA = EC,
the ∠EAC = the ∠ECA;
∴ the whole angle BAC = the two angles ABC, ACB. [*Ax.* 2.
But FAC, the exterior angle of the triangle ABC,
= the two angles ABC, ACB; [I. 32.
∴ the ∠BAC = the ∠FAC, [*Axiom* 1.
and therefore each of them is a right ∠; [I. *Definition* 10.
∴ the ∠ in a semicircle BAC is a right ∠.

(2) Because the two angles ABC, BAC of the triangle ABC are together less than two right angles, [I. 17.
and that BAC has been shewn to be a right $\angle$,
$\therefore$ the $\angle$ABC is less than a right $\angle$;
$\therefore$ the $\angle$ in a segment ABC, greater than a semicircle, is less than a right $\angle$.

(3) Because ABCD is a quadrilateral figure in a circle, any two of its opposite angles together = two right angles; [III. 22.
$\therefore$ the angles ABC, ADC together = two right angles.
But the $\angle$ABC has been shewn to be less than a right $\angle$.
$\therefore$ the $\angle$ADC is greater than a right $\angle$;
$\therefore$ the $\angle$ in a segment ADC, less than a semicircle, is greater than a right $\angle$.

Wherefore, *the angle, etc.* [Q.E.D.

Corollary. *If one angle of a triangle be equal to the other two, it is a right angle.*
For the angle adjacent to it is equal to the same two angles; [I. 32.
and when the adjacent angles are equal, they are right angles.
[I. *Definition* 10.

EXERCISES.

****1.** Right-angled triangles are described on the same hypotenuse; shew that the angular points opposite the hypotenuse all lie on a circle described on the hypotenuse as diameter.

2. The circles described on the sides of any triangle as diameters, will intersect on the base.

3. AOB and COD are two perpendicular diameters of a circle. If P be any point on the circumference, prove that CP and DP are the internal and external bisectors of the angle APB.

4. Chords AB, CD of a circle cut one another at right angles; the sum of the opposite arcs AC and BD is a semicircle.

5. AB is the diameter of a circle whose centre is C, and DCE is a sector having the arc DE constant; join AE, BD intersecting at P; shew that the angle APB is constant.

6. On the side AB of any triangle ABC as diameter a circle is described; EF is a diameter parallel to BC; shew that the straight lines EB and FB bisect the interior and exterior angles at B.

7. If AD, CE be drawn perpendicular to the sides BC, AB of a triangle ABC, and DE be joined, shew that the angles ADE and ACE are equal. [A, E, D, C lie on a circle.]

8. If two circles ABC, ABD intersect at A and B, and AC, AD be two diameters, shew that the straight line CD will pass through B.

9. If O be the centre of a circle and OA a radius, and a circle be described on OA as diameter, the circumference of this circle will bisect any chord of the exterior circle drawn from A.

10. If from the angles at the base of any triangle perpendiculars are drawn to the opposite sides, produced if necessary, the straight line joining the points of intersection will be bisected by a perpendicular drawn to it from the centre of the base.

11. If two chords of a circle meet at right angles within or without a circle, the squares on their segments are together equal to the squares on the diameter.

[Let the chords be AB, CD meeting within the circle at O and let AA′ be the diameter through A. Then

$$\angle CAB = \text{rt.} \angle - ACD = \text{rt.} \angle - AA'D = \angle DAA';$$
$$\therefore\ DA' = BC;$$
$$\therefore\ AO^2 + OD^2 + BO^2 + OC^2 = AD^2 + BC^2 = AD^2 + DA'^2 = AA'^2.$$

Similarly if the chords cut without the circle.]

12. AB is a diameter of a circle and C a given point in AB; find a point in the circumference at which both AC and CB will subtend half a right angle.

[Join C to the middle point D of the arc AB; the required point is the point where CD meets the circle again.]

13. Circles are described on the sides of a quadrilateral as diameters; prove that the common chord of two of these circles which are adjacent is parallel to the common chord of the other two.

14. ABCD is a quadrilateral figure inscribed in a circle; if the bisectors of two of its opposite angles meet the circle in E and F, then EF is a diameter.

15. Divide a straight line into two parts such that the rectangle contained by the two parts may be equal to a given square. When is this impossible? [Use II. 5.]

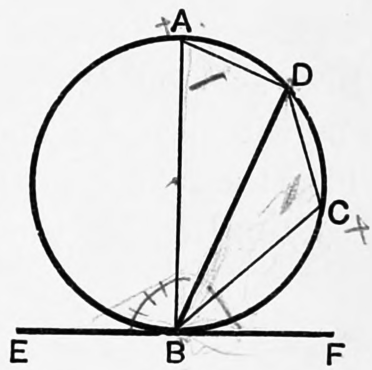
PROPOSITION 32. THEOREM.

If a straight line touch a circle, and from the point of contact a chord be drawn, the angles which this chord makes with the tangent shall be equal to the angles which are in the alternate segments of the circle.

Let the straight line EF touch the circle ABCD at the point B, and from B let the chord BD be drawn, cutting the circle: then

(1) *the angle* DBF *shall be equal to the angle in the segment* BAD, *and*

(2) *the angle* DBE *shall be equal to the angle in the segment* BCD.



Construction. From B draw BA at right angles to EF, [I. 11.
and take any point C in the arc BD.

Join AD, DC, CB.

Proof. (1) Because BA is drawn at right angles to the tangent EF at its point of contact B, [*Constr.*
∴ the centre of the circle is in BA; [III. 19.
∴ the ∠ADB, being in a semicircle, is a right ∠; [III. 31.
∴ the other two angles BAD, ABD together = a right ∠. [I. 32.
But ABF is also a right angle; [*Constr.*
∴ the ∠ABF = the angles BAD, ABD.
From each of these equals take the common ∠ABD;
∴ the remaining ∠DBF = the remaining ∠BAD, [*Ax.* 3.
which is in the alternate segment of the circle.

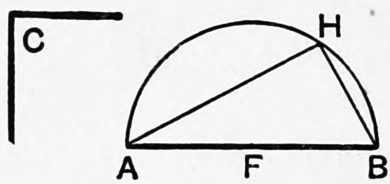
(2) Because ABCD is a quadrilateral in a circle, the angles BAD, BCD together = two right angles. [III. 22.
But the angles DBF, DBE together = two right angles; [I. 13.
∴ the angles DBF, DBE together = the angles BAD, BCD.
And the ∠DBF has been shewn equal to the ∠BAD;
∴ the remaining ∠DBE = the remaining ∠BCD, [*Ax.* 3.
which is in the alternate segment of the circle.

Wherefore, *if a straight line, etc.* [Q.E.D.

PROPOSITION 33. PROBLEM.

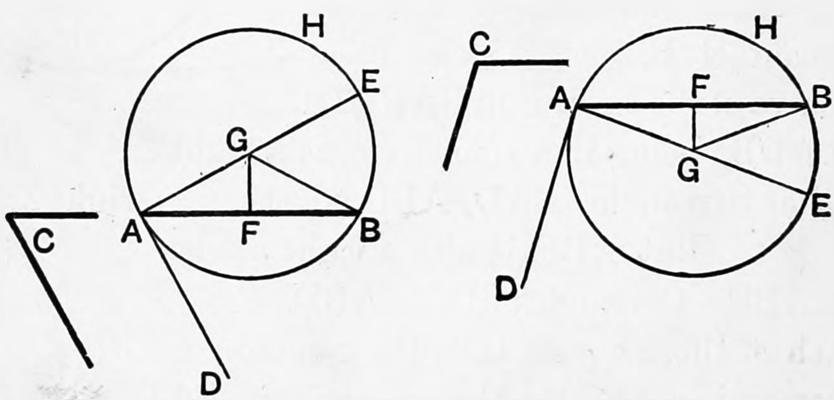
On a given straight line to describe a segment of a circle, containing an angle equal to a given rectilineal angle.

Let AB be the given straight line, and C the given rectilineal angle:
it is required to describe on AB *a segment of a circle containing an angle equal to* C.



CASE I. First, let the ∠C be a right ∠.
Bisect AB at F, [I. 10.
and with centre F and radius FB describe the semicircle AHB.
Then the ∠AHB in a semicircle is equal to the right ∠C. [III. 31.

CASE II. Let the ∠C be not a right ∠.



Construction. At A make the ∠BAD equal to the ∠C; [I. 23.
from A draw AE at right angles to AD; [I. 11.
bisect AB at F; [I. 10.
from F draw FG at right angles to AB to meet AE in G. [I. 11.
Join GB.

Proof. In the triangles AFG, BFG,

because { AF = BF,
and FG is common,
and the right ∠AFG = the right ∠BFG; [I. *Ax.* 11.

∴ the base AG = the base BG, [I. 4.

and therefore the circle described from the centre G, with radius GA, will pass through the point B.

Let this circle be described; and let it be AHB.

The segment AHB shall contain an angle equal to the given rectilineal angle C.

Because AD is drawn at right angles to the diameter AE,

∴ AD touches the circle. [III. 16. *Corollary.*

Also, because AB is drawn from the point of contact A,

the ∠DAB = the ∠ in the alternate segment AHB. [III. 32.

But the ∠DAB = the ∠C; [*Construction.*

∴ the ∠ in the segment AHB = the ∠C. [*Axiom* 1.

Wherefore, *on the given straight line* AB, *the segment* AHB *of a circle has been described, containing an angle equal to the given angle* C. [Q.E.F.

EXERCISES ON PROPOSITION 32.

1. Prove the converse of III. 32. [See Notes, page 327.]

****2.** If a tangent to a circle is parallel to a chord, the point of contact of the tangent will be the middle point of the arc cut off by the chord.

3. Two circles touch each other externally. Prove that any straight line drawn through the point of contact cuts off similar segments from the two circles.

4. If two circles intersect one another, prove that each common tangent subtends at either common point angles that are equal or supplementary.

5. B is a point in the circumference of a circle whose centre is C; PA, a tangent at any point P, meets CB produced at A, and PD is drawn perpendicular to CB; shew that the straight line PB bisects the angle APD.

6. Two circles intersect at A and B, and through P any point in the circumference of one of them the chords PA and PB are drawn to cut the other circle at C and D; shew that CD is parallel to the tangent at P.

7. If from any point in the circumference of a circle a chord and tangent be drawn, the perpendiculars dropped on them from the middle point of the intercepted arc are equal.

8. AB is a chord and AD a tangent to a circle at A. DPQ is a straight line parallel to AB, meeting the circle in P and Q. Prove that the triangles PAD, QAB are equiangular.

9. On a straight line AB as base, and on the same side of it are described two segments of circles; P is any point in the circumference of one of the segments, and the straight line BP cuts the circumference of the other segment at Q; shew that the angle PAQ is equal to the angle between the tangents at A.

10. C is the centre of a circle; CA, CB are two radii at right angles; from B any chord BP is drawn cutting CA at N: a circle being described round ANP, shew that it will be touched by BA.

11. AB and CD are parallel straight lines, and the straight lines which join their extremities intersect at E: shew that the circles described round the triangles ABE, CDE touch one another.

12. If the centres of two circles which touch each other externally be fixed, the common tangent of those circles will touch another circle of which the straight line joining the fixed centres is the diameter.

EXERCISES ON PROPOSITION 33.

1. Construct a triangle, having given the base, the vertical angle, and the point in the base on which the perpendicular falls.

[On the given base AB describe a segment of a circle containing the given vertical angle; by the converse of III. 22 the vertex lies somewhere on this segment. Through the given point in the base draw a straight line perpendicular to the base; this will meet the segment in the required vertex. There are two solutions, since two segments can be drawn, one on each side of AB.]

2. Construct a triangle, having given the base, the vertical angle, and the altitude.

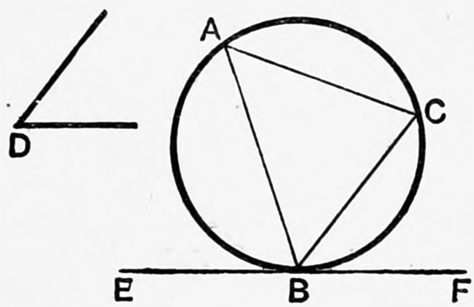
3. Construct a triangle, having given the base, the vertical angle, and the length of the straight line drawn from the vertex to the middle point of the base.

4. Having given the base and the vertical angle of a triangle, shew that the triangle will be greatest when it is isosceles.

PROPOSITION 34. PROBLEM.

From a given circle to cut off a segment containing an angle equal to a given rectilineal angle.

Let ABC be the given circle, and D the given angle: *it is required to cut off from the circle* ABC *a segment containing an angle equal to* D.



Construction. Draw the straight line EF touching the circle ABC at B; [III. 17.
and at B make the ∠FBC equal to the ∠D. [I. 23.
The segment BAC shall contain an ∠ equal to D.

Proof. Because EF touches the circle ABC, and BC is drawn from the point of contact B; [*Construction.*
∴ the ∠FBC = the ∠ in the alternate segment BAC. [III. 32.
But the ∠FBC = the ∠D; [*Construction.*
∴ the ∠ in the segment BAC = the ∠D. [*Axiom* 1.

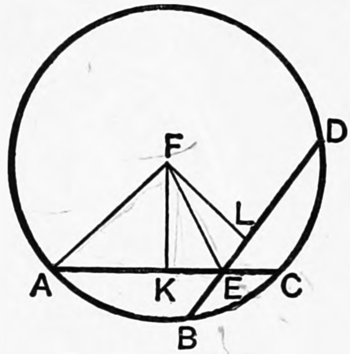
Wherefore, *from the given circle* ABC, *the segment* BAC *has been cut off, containing an angle equal to the given angle* D. [Q.E.F.

ℓe

PROPOSITION 35. THEOREM.

If two chords of a circle cut one another within the circle, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.

Let the two chords AC, BD cut one another at the point E within the circle;
the rectangle AE, EC *shall be equal to the rectangle* BE, ED.



Construction. CASE I. Find the centre F, and, if it be not in AC or BD, join FE.

Draw FK perpendicular to AC and FL perpendicular to BD. Join FA.

Proof. Since AC is bisected at K and divided unequally at E,
∴ the rect. AE, EC together with the square on KE
= the square on AK. [II. 5.

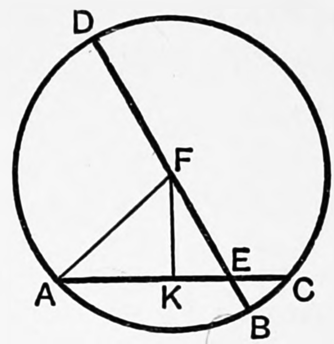
To each add the square on FK,
∴ the rect. AE, EC, together with the squares on FK, KE,
= the squares on AK, KF.

But, since the angles at K are right angles, [*Construction.*
the squares on FK, KE = the square on FE, [I. 47.
and the squares on AK, KF = the square on FA; [I. 47.
∴ the rect. AE, EC together with the square on FE
= the square on FA.

Similarly, the rect. BE, ED together with the square on FE
= the square on FD, that is, the square on FA;
∴ the rect. AE, EC with the square on FE
= the rect. BE, ED with the square on FE.

Take away the common square on FE;
∴ the rect. AE, EC = the rect. BE, ED.

CASE II. Next, let the centre F be on one of the chords BD, and let BD be not perpendicular to AC.



Then since BD is bisected at F and divided unequally at E,

∴ the rect. BE, ED with the square on EF = the square on FB, [II. 5.

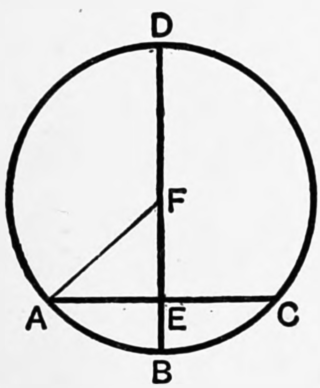
that is, the square on FA.

Also, as in Case I.,

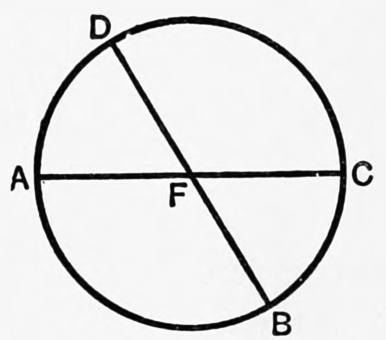
the rect. AE, EC with the square on FE = the square on FA;

∴ as before, the rect. AE, EC = the rect. BE, ED.

CASE III. Next, let F be on BD, and let BD be perpendicular to AC. Then BD will bisect AC;



CASE III.



CASE IV.

∴ the rect. AE, EC = the square on AE,

and ∴ the rect. AE, EC, together with the square on EF,

= the squares on AE, EF,

that is, = the square on FA, that is, as in the second case,

= the rect. BE, ED with the square on EF,

and thus the rect. AE, EC = the rect. BE, ED

CASE IV. Lastly, let both chords pass through the centre F. In this case the lines AF, FC, BF, FD are all equal;

∴ the rect. AF, FC = the rect. BF, FD;

for each is equal to the square on the radius of the circle.

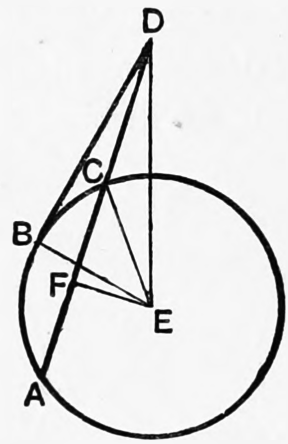
Wherefore, *if two chords, etc.*

PROPOSITION 36. THEOREM.

If from any point without a circle two straight lines be drawn, one of which cuts the circle, and the other touches it; the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, shall be equal to the square on the line which touches it.

From any point D without the circle ABC let there be drawn two straight lines DCA, DB, of which DCA cuts the circle and DB touches it:

the rectangle AD, DC *shall be equal to the square on* DB.

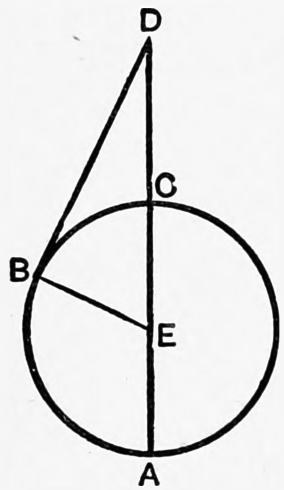


Construction. CASE I. Let DCA not pass through the centre of the circle ABC; find the centre E; [III. 1.
from E draw EF perpendicular to AC; [I. 12.
and join EB, EC, ED.

Proof. Because EF drawn from the centre cuts the chord AC at right angles at F, ∴ AF = FC. [III. 3.
And because AC is bisected at F, and produced to D, the rectangle AD, DC, together with the square on FC,
= the square on FD. [II. 6.
To each of these equals add the square on FE;
∴ the rect. AD, DC, with the squares on CF, FE,
= the squares on DF, FE. [*Axiom* 2.
But, because CFE is a right angle, the squares on CF, FE
= the square on CE, [I. 47.
that is, = the square on BE;

and the squares on DF, FE = the square on DE;
∴ the rect. AD, DC, together with the square on BE,
= the square on DE.
that is, = the squares on DB, BE,
because EBD is a right angle. [I. 47.
Take away the common square on BE;
then the remaining rect. AD, DC = the square on DB. [*Axiom* 3.

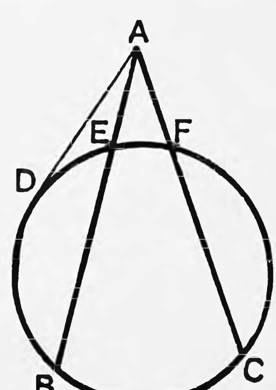
CASE II. If DCA passes through E, join EB.



Since AC is bisected at E and produced to D,
the rect. AD, DC, with the square on EC,
= the square on ED, [II. 6.
that is, = the sqs. on EB, BD, since EBD is a right ∠. [I. 47.
But the square on EC = the square on EB;
∴ the rect. AD, DC = the square on BD.

Wherefore, *if from any point, etc.* [Q.E.D.

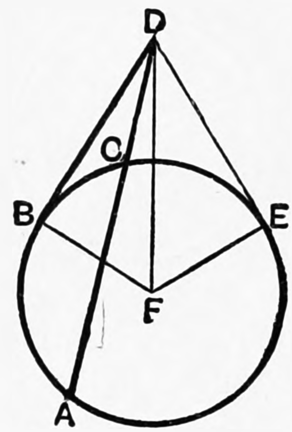
Corollary. If from any point without a circle there be drawn secants AEB, AFC, the rectangles contained by the whole secants and the parts of them without the circles are equal, that is, the rectangle BA, AE = the rectangle CA, AF; for each of them is equal to the square on the tangent AD.



PROPOSITION 37. THEOREM.

If from any point without a circle there be drawn two straight lines, one of which cuts the circle, and the other meets it, and if the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, be equal to the square on the line which meets the circle, the line which meets the circle shall be a tangent.

Let any point D be taken without the circle ABC, and from it let two straight lines DCA, DB be drawn, of which DCA cuts the circle, and DB meets it; and let the rectangle AD, DC be equal to the square on DB:
DB *shall touch the circle.*



Construction. Draw the straight line DE, touching the circle ABC; [III. 17.

find F the centre, [III. 1.

and join FB, FD, FE.

Proof. Because DE touches the circle ABC, and DCA cuts it,

the rect. AD, DC = the square on DE. [III. 36.

But the rect. AD, DC = the square on DB; [*Hypothesis.*

∴ the square on DE = the square on DB. [*Axiom* 1.

Hence, in the triangles DBF, DEF,

because { DE = DB,
and EF = BF, I. *Definition* 15.
and the base DF is common,

∴ the ∠DEF = the ∠DBF. [I. 8.

But DEF is a right ∠, since DE is a tangent ; [*Constr.* and III. 18.
∴ also DBF is a right ∠.
And BF, if produced, is a diameter ;
∴ DB touches the circle ABC. [III. 16, *Corollary.*
Wherefore, *if from a point, etc.* [Q.E.D.

EXERCISES.

****1.** Prove the following converse of III. 35 and III. 36, Cor. : *If two straight lines* AB, CD *intersect in* O *and the rectangle* AO, OB = *the rectangle* CO, OD, *the four points* A, B, C, D *lie on a circle.*

[For if the circle passing through A, B, C do not pass through D, let it meet CD in E ; then, by III. 35 or III. 36 Cor. (according as O is within or without CD) the rectangle CO, OE = the rectangle AO, OB = the rectangle CO, OD (Hyp.), ∴ OE = OD. Hence E coincides with D, and thus the circle through A, B, C passes through D.]

****2.** If two circles cut one another, the tangents drawn to the two circles from any point in the common chord produced are equal.

****3.** Two circles intersect. Shew that their common chord produced bisects their common tangent.

4. If AD, CE are drawn perpendicular to the sides BC, AB of a triangle ABC, shew that the rectangle contained by BC and BD is equal to the rectangle contained by BA and BE.

5. If through any point in the common chord of two circles which intersect one another, there be drawn any two other chords, one in each circle, their four extremities all lie on the circumference of a circle.

6. From a given point as centre, describe a circle cutting a given straight line in two points, so that the rectangle contained by their distances from a fixed point in the straight line may be equal to a given square.

[The radius of the circle is such that its square is equal to the difference of the square on the line joining the two given points and the given square.]

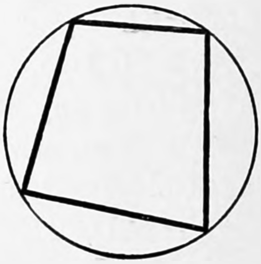
7. A series of circles intersect each other, and are such that the tangents to them from a fixed point are equal. Shew that the straight lines joining the two points of intersection of each pair will pass through this point.

BOOK IV.

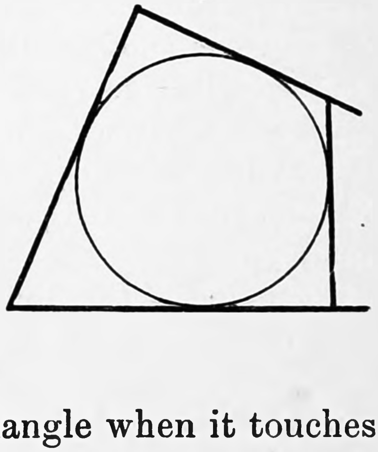
DEFINITIONS.

1. A rectilineal figure is said to be **inscribed** in another rectilineal figure when all the angles of the inscribed figure are on the sides of the figure in which it is inscribed, each on each.

2. In like manner a figure is said to be **described about,** or **circumscribed about,** another figure when all the sides of the circumscribed figure pass through the angular points of the figure about which it is described, each through each.



3. A rectilineal figure is said to be **inscribed** in a circle when all the angles of the inscribed figure are on the circumference of the circle.



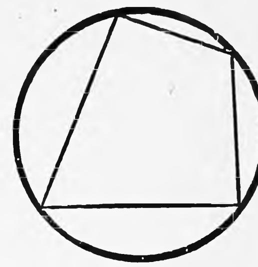
4. A rectilineal figure is said to be described about a circle when each side of the circumscribed figure touches the circumference of the circle.

5. In like manner a circle is said to be inscribed in a rectilineal figure when the circumference of the circle touches each side of the figure.

Such a circle is often called the **in-circle** and its centre the **in-centre.**

[A circle is said to be **escribed** to a triangle when it touches one side of the triangle and the other two sides produced. For a figure, see page 181.]

6. A circle is said to be described about a rectilineal figure when the circumference of the circle passes through all the angular points of the figure about which it is described.



Such a circle is often called the **circum-circle** and its centre the **circum-centre.**

7. A straight line is said to be placed in a circle when the extremities of it are on the circumference of the circle.

8. A polygon is a rectilineal figure contained by more than four straight lines. [I. *Def.* 22.

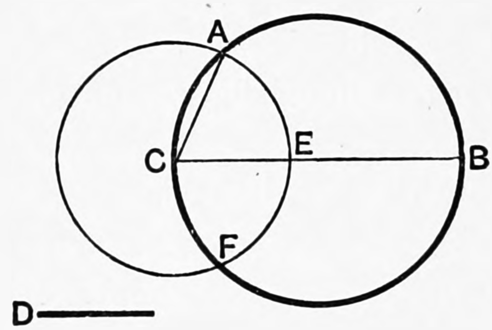
A polygon of five sides is called a **pentagon** (Gk. πέντε).
,, six ,, ,, **hexagon** (Gk. ἕξ).
,, seven ,, ,, **heptagon** (Gk. ἑπτά).
,, eight ,, ,, an **octagon** (Gk. ὀκτώ).
,, ten ,, ,, a **decagon** (Gk. δέκα).
,, twelve ,, ,, **dodecagon** (Gk. δώδεκα).
,, fifteen ,, ,, **quindecagon** (Lat. *quinque*; Gk. δέκα).

9. A **regular polygon** is one which has all its sides equal and also all its angles equal, *i.e.* it is equilateral and equi-angular.

Proposition 1. Theorem.

In a given circle, to draw a chord equal to a given straight line, which is not greater than the diameter of the circle.

Let ABC be the given circle, and D the given straight line, not greater than the diameter of the circle:
it is required to draw a chord of the circle ABC *equal to* D.



Construction. Draw BC, a diameter of the circle ABC. Then, if BC = D, the thing required is done; for, in the circle ABC, a straight line is placed equal to D.
But, if it is not, BC is greater than D. [*Hypothesis.*
Make CE equal to D, [I. 3.
and with centre C and radius CE describe the circle AEF and join CA.

Proof. Because C is the centre of the circle AEF,

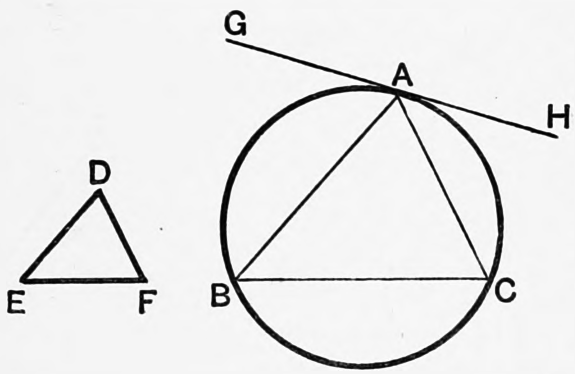
CA = CE; [I. *Definition* 15.
but CE = D; [*Construction.*
∴ CA = D. [*Axiom* 1.

Wherefore, *in the circle* ABC, *a straight line* CA *is placed equal to the given straight line* D, *which is not greater than the diameter of the circle.* [Q.E.F.

PROPOSITION 2. PROBLEM.

In a given circle, to inscribe a triangle equiangular to a given triangle.

Let ABC be the given circle, and DEF the given triangle: *it is required to inscribe in the circle* ABC *a triangle equiangular to* DEF.



Construction. At any point A on the circumference draw the tangent GAH; [III. 17.
at A make the ∠HAC equal to the ∠DEF, [I. 23.
and also make the ∠GAB equal to the ∠DFE;
join BC. ABC shall be the triangle required.

Proof. Because GAH touches the circle ABC, and AC is drawn from the point of contact A; [*Construction.*
∴ the ∠HAC = the ∠ABC in the alternate segment. [III. 32.
But the ∠HAC = the ∠DEF; [*Construction.*
∴ the ∠ABC = the ∠DEF. [*Axiom* 1.

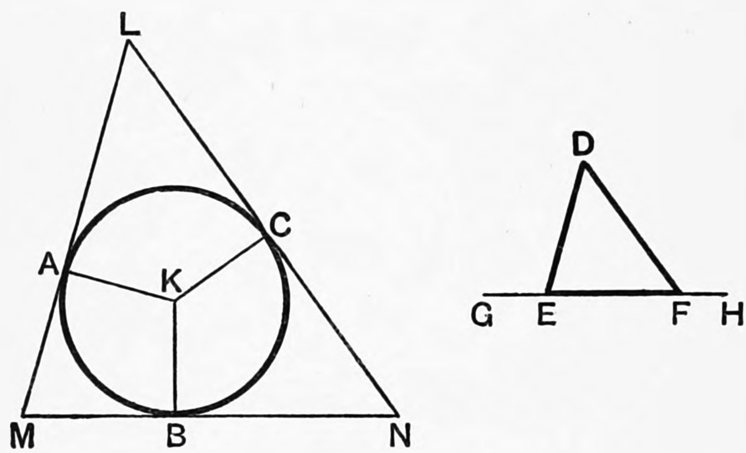
Similarly, the ∠ACB = the ∠DFE;
∴ the third ∠BAC = the third ∠EDF. [I. 32, *Axioms* 11 and 3.

Wherefore *the triangle* ABC *is equiangular to the triangle* DEF, *and it is inscribed in the circle* ABC. [Q.E.F.

Proposition 3. Problem.

About a given circle, to describe a traingle equiangular to a given triangle.

Let ABC be the given circle, and DEF the given triangle: *it is required to describe a triangle about the circle* ABC *equiangular to the* △DEF.



Construction. Produce EF both ways to G and H;
find K, the centre of the circle ABC; [III. 1.
from K draw any radius KB;
at K make the ∠BKA equal to the ∠DEG,
and the ∠BKC equal to the ∠DFH; [I. 23.
and through A, B, C draw the straight lines LAM, MBN, NCL at right angles to KA, KB, and KC respectively. [I. 11.
LMN shall be the triangle required.

Proof. Because LM, MN, NL are drawn perpendicular to the radii KA, KB, KC through their extremities, [*Construction.*
therefore LM, MN, NL all touch the circle,
and LMN is a △ described about the circle.
Also because the four angles of the figure AMBK are together equal to four right angles,
for it can be divided into two triangles,
and two of them KAM, KBM are right angles,
∴ the other two AKB, AMB are together equal to two right angles. [*Axiom* 3.

But the angles DEG, DEF together = two right angles; [I. 13.
∴ the angles AKB, AMB = the angles DEG, DEF;
of which the ∠AKB = the ∠DEG; [*Construction.*
∴ the remaining ∠AMB = the remaining ∠DEF. [*Ax.* 3.
Similarly, the angles LNM and DFE may be shewn to be equal;
∴ the third ∠MLN = the third ∠EDF. [I. 32, *Axioms* 11 and 3.

Wherefore *the triangle* LMN *is equiangular to the triangle* DEF, *and it is described about the circle* ABC. [Q.E.F.

EXERCISES.

1. Place a chord in a given circle equal to a given straight line, so that it shall be parallel to another given straight line.

2. Place a chord of given length in a given circle, so that it may pass through a given point within or without the circle. When is this impossible?

3. Inscribe in a circle a triangle MNP whose sides are parallel to three given straight lines.

4. Two triangles are circumscribed to a given circle, each of them being equiangular to a given triangle; prove that the triangles are equal in all respects.

5. Any rectilinear figure ABCDE is inscribed in a circle; the arcs AB, BC, CD, DE, EA are bisected, and tangents drawn at the points of bisection; shew that the resulting figure is equiangular to ABCDE.

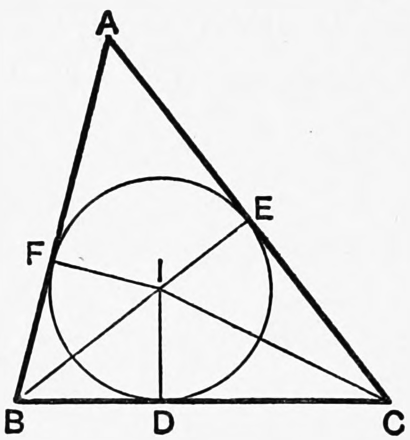
6. Prove that the area of an equilateral triangle inscribed in a circle is one-quarter that of the equilateral triangle circumscribed to the circle.

[Let ABC be an equilateral triangle inscribed in a given circle; at A, B, C draw tangents. Let the tangents at B, C meet in A′; those at C, A in B′; and those at A, B in C′. Then A′B′C′ is an equilateral △ described about the circle, and it is easy to prove that the △s A′CB, B′AC, C′BA and ABC are all equal.]

PROPOSITION 4. PROBLEM.

To inscribe a circle in a given triangle.

Let ABC be the given triangle:
it is required to inscribe a circle in the triangle ABC.



Construction. Bisect the angles ABC, ACB by the straight lines BI, CI, meeting one another at the point I; [I. 9.
and from I draw ID, IE, IF perpendiculars to BC, CA, AB. [I. 12.

Proof. In the triangles DBI, FBI,

because $\begin{cases} \text{the } \angle DBI = \text{the } \angle FBI, \\ \text{and the right } \angle BDI = \text{the right } \angle BFI, \\ \text{and the side BI is common;} \end{cases}$ [*Construction.*

$\therefore$ ID = IF. [I. 26.

For the same reason ID = IE;

$\therefore$ IE = IF; [*Axiom* 1.

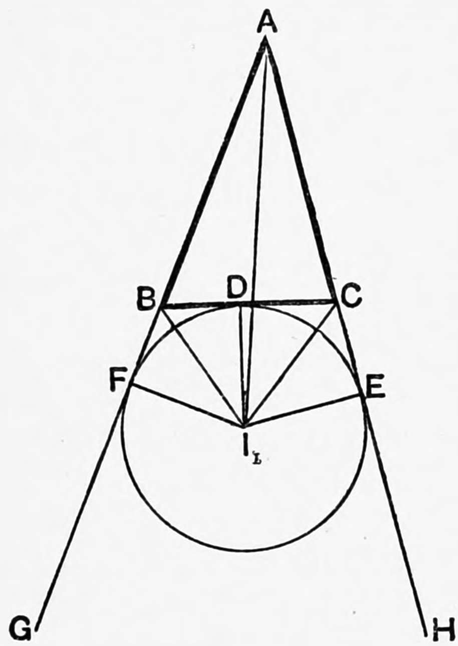
$\therefore$ the three straight lines ID, IE, IF are equal and the circle described with centre I, and radius equal to any one of them, will pass through the extremities of the other two;
and it will touch the straight lines BC, CA, AB, because the angles at the points D, E, F are right angles; [III. 16, *Cor.* 1.
$\therefore$ the circle DEF is inscribed in the triangle ABC.

Wherefore *a circle has been inscribed in the given triangle.* [Q.E.F.

ADDITIONAL PROPOSITION.

To draw an escribed circle of a given triangle.

Let ABC be the given triangle:
it is required to draw the escribed circle which is opposite to the angle A, *that is, which touches* BC *externally.*



Construction. Produce AB and AC to G and H;
bisect the exterior angles GBC, HCB by the straight lines BI_1, CI_1, which meet at I_1;
draw I_1D, I_1E, and I_1F perpendicular to BC, CA, and AB respectively.

Proof. In the triangles I_1BD, I_1BF,

because $\left\{\begin{array}{l} \text{the } \angle I_1BD = \text{the } \angle I_1BF, \\ \text{and the right } \angle I_1DB = \text{the right } \angle I_1FB, \\ \text{and the side } I_1B \text{ is common;} \end{array}\right.$ [*Construction.*

$\therefore$ the base I_1D = the base I_1F. [I. 26.
Similarly, it may be shewn that $I_1D = I_1E$;
therefore I_1D, I_1E, and I_1F are equal, and a circle described with centre I_1, and radius equal to either of these three, will pass through the extremities of the other two.
Also, since the angles at D, E, F are right angles, this circle will touch BC and AB, AC produced, and will therefore be the circle required.

Corollary. Since this circle touches AE, AF, the angles I_1AE and I_1AF will be equal, [III. 17, *Corollary.*
and I_1A will therefore bisect the angle BAC.

EXERCISES.

****1.** In the figure of IV. 4, prove that AI bisects the angle BAC, and hence that the bisectors of the angles of a triangle meet in a point.

****2.** In the figure of IV. 4, prove that $BD + CA = CE + AB = AF + BC$ = the semi-perimeter of the triangle ABC.

[$BD = BF$, $CD = CE$, $AE = AF$ (III. 17, *Cor.*);

$\therefore$ sum of twice BD, CE, AE = sum of BD, BF, CE, CD, AE, AF
= sum of BC, CA, AB;

$\therefore$ twice BD and twice AC = perimeter, etc.]

3. The circle inscribed in a triangle ABC touches the sides in the points D, E, and F. Prove that the angles of the triangle DEF are equal to the complements of half the angles of the triangle ABC, and hence that the triangle DEF is always acute-angled.

4. Without producing two straight lines to meet, find that straight line which would bisect the angle between them.

[Suppose BK, CL to be the two lines; draw any straight line BC to meet them; bisect $\angle^s$ at B, C by straight lines BI, CI meeting in I; draw IF, IE perpendicular to BK, CL and bisect FIE; this bisecting line will be the required line.]

5. With the vertices A, B, C of a triangle as centres draw three circles, each of which touches the other two.

6. A circle is inscribed in a triangle ABC, and a triangle is cut off at each angle by a tangent to the circle. Shew that the sides of the three triangles so cut off are together equal to the sides of ABC.

7. If the circle inscribed in a triangle ABC touch the sides AB, AC at the points D, E, prove that the middle point of the arc DE is the centre of the circle inscribed in the triangle ADE.

8. Find the centre of a circle cutting off three equal chords from the sides of a triangle.

The escribed circle opposite the angle A of a triangle touches the sides in the points D_1, E_1, F_1 and the inscribed circle touches them in D, E, and F; prove that

****9.** $AE_1 = AF_1$ = the semi-perimeter of the $\triangle$ ABC.

10. $BD = CD_1$.

11. DD_1 = the difference between the sides AB and AC.

12. Two sides of a triangle whose perimeter is constant are given in position; prove that the third side always touches a certain circle.

I is the in-centre and I_1, I_2, I_3 the centres of the escribed circles of the triangles ABC; prove that

****13.** AII_1, BII_2, and CII_3 are straight lines. [See App. Art. 48.]

****14.** I_2AI_3, I_3BI_1, and I_1CI_2 are straight lines.

****15.** AII_1 is perpendicular to I_2I_3, etc.

16. I, B, C, I_1 lie on a circle.

17. I_2, I_3, B, C lie on a circle.

18. the triangles BCI_1, CAI_2, and ABI_3 are equiangular.

19. If the escribed circles opposite to the angles A, B, C of a triangle touch the sides BC, CA, AB in D_1, D_2, D_3, prove that $AD_2 = BD_1$, $BD_3 = CD_2$, and $CD_1 = AD_3$.

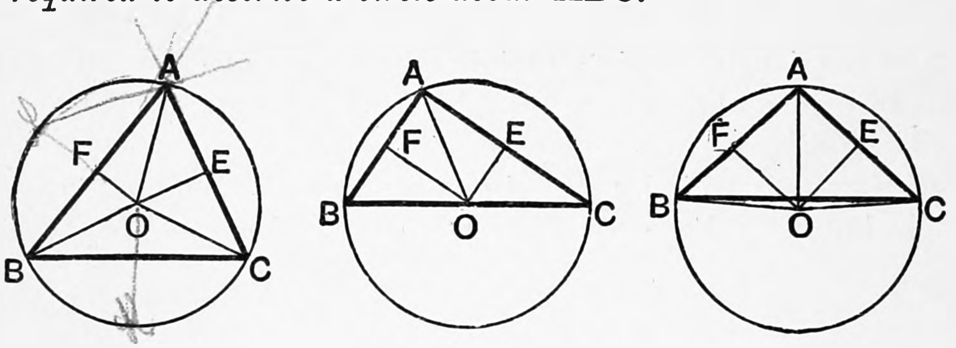
20. The triangle formed by joining the centres of the escribed circles of the triangle and that formed by joining the points of contact of the inscribed circle are equiangular.

21. How many circles can, in general, be drawn to touch three straight lines? What are the exceptional cases?

Proposition 5. Problem

To describe a circle about a given triangle.

Let ABC be the given triangle:
it is required to describe a circle about ABC.



Construction. Bisect AB, AC at the points F, E; [I. 10.
from these points draw FO, EO at right angles to AB, AC; [I. 11.

FO, EO, produced, will meet one another;
for if they do not meet they are parallel;
therefore AB, AC, which are at right angles to them, are parallel, which is absurd:
let them meet at O, and join OA;
also, if O be not in BC, join BO, CO.

Proof. In the triangles AFO, BFO,

because $\begin{cases} \text{AF} = \text{BF}, & \text{[\textit{Construction}.} \\ \text{and FO is common,} & \\ \text{and the right } \angle\text{AFO} = \text{the right } \angle\text{BFO}; & \end{cases}$

∴ the base OA = the base OB. [I. 4.

Similarly, it may be shewn that OC = OA;

∴ OB = OC, [*Axiom* 1.

and OA, OB, OC are all equal.

Therefore the circle described with centre O, and radius equal to any one of them, will pass through the extremities of the other two, and will be described about the triangle ABC.

Wherefore *a circle has been described about the given triangle.*

[Q.E.F.

Corollary. It is clear that

(i.) when the centre of the circle falls within the triangle, each of its angles is less than a right angle, each of them being in a segment greater than a semicircle;

(ii.) when the centre is in one of the sides of a triangle, the $\angle$ opposite to this side, being in a semicircle, is a right $\angle$; and

(iii.) when the centre falls without the triangle, the $\angle$ opposite to the side beyond which it is, being in a segment less than a semicircle, is greater than a right $\angle$. [III. 31.

Therefore, conversely, *if the given triangle be acute-angled, the centre of the circle falls within it; if it be a right-angled triangle, the centre is in the side opposite to the right angle; and if it be an obtuse-angled triangle, the centre falls without the triangle, beyond the side opposite to the obtuse angle.*

EXERCISES.

****1.** In IV. 5, shew that the perpendicular from O on BC will bisect BC, and hence that the straight lines which bisect at right angles the sides of a triangle all meet in a point.

****2.** If the inscribed and circumscribed circles of a triangle be concentric, shew that the triangle must be equilateral.

The perpendicular from O on BC meets the circum-circle in K and L (K being on the opposite side of BC from A). Prove that

3. $\angle BOK = \angle COK = A$.

4. AK and AL bisect the interior and exterior angles at A.

****5.** K is the centre of the circum-circle of BIC, where I is the in-centre.

6. The angle between the radius of the circum-circle passing through the vertex A of a triangle ABC, and the perpendicular from A upon BC is equal to the difference of the base angles of the triangle.

7. O is the centre of the circle circumscribing a triangle ABC; D, E, F the feet of the perpendiculars from A, B, C on the opposite

sides. Shew that OA, OB, OC are respectively perpendicular to EF, FD, DE.

8. In an equilateral triangle, prove that the radius of the circum-circle is twice that of the in-circle, and the radius of the escribed circle is three times that of the in-circle.

[Let O be the in- and circum-centre of the equilateral $\triangle$ ABC and O_1 the e-centre opposite A; let AD be $\perp^r$ to BC and P the middle point of OO_1. Then OCO_1 is a rt. $\angle$ and $\therefore$ P is the circum-centre of $\triangle\, OCO_1$; $\therefore$ $PO_1 = PC = PO$; $\therefore$ $\angle PCO = \angle POC = 2\angle ACO = 2\angle OCD$ $= 2$ complement of $\angle DCO_1 = 2\angle DO_1C = \angle DPC$, and $\therefore$ POC is equilateral; $\therefore$ $OC = OP = 2\,OD$, and $O_1D = O_1P + PD = 2\,OD + OD = 3\,OD$.]

9. The side BC of a triangle ABC is produced to D so that the triangles ABD, ACD are equiangular. Prove that AD touches the circum-circle of the triangle ABC.

10. A quadrilateral ABCD is inscribed in a circle, and AD, BC are produced to meet at E. Shew that the circle described about the triangle ECD will have the tangent at E parallel to AB.

11. If DE be drawn parallel to the base BC of a triangle ABC to meet the sides in D and E, shew that the circles described about the triangles ABC and ADE have a common tangent.

12. The diagonals of a given quadrilateral ABCD intersect at O. Shew that the centres of the circles described about the triangles OAB, OBC, OCD, ODA are at the angular points of a parallelogram.

[Each of its sides is perpendicular to one of the diagonals of ABCD.]

13. The opposite sides of a quadrilateral inscribed in a circle are produced to meet in P and Q, and about the triangles so formed without the quadrilateral circles are described, which meet in R. Prove that PRQ is a straight line.

14. Three circles whose centres are A, B, and C touch one another externally in D, E, and F. Prove that the in-circle of the triangle ABC is the circum-circle of the triangle DEF.

[Draw the common tangents at D, E to meet in O. Then shew that OFA is a right $\angle$ so that the common tangent at F goes through O. Then prove that $OA = OB = OC$ and $OD = OE = OF$.]

****15.** The four circles each of which passes through the centres of three of the four circles touching the sides of a triangle are equal to one another.

16. If L, M, N be any three points on the sides BC, CA, AB of a triangle, prove that the circles which circumscribe the triangles MAN, NBL, LCM meet in a point.

PROPOSITION 6. PROBLEM.

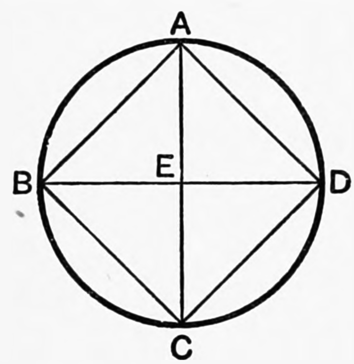
To inscribe a square in a given circle.

Let ABCD be the given circle:
it is required to inscribe a square in ABCD.

Construction. Find the centre E of the circle, and draw two diameters AC, BD at right angles to one another; [III. 1, I. 11.

and join AB, BC, CD, DA.

The figure ABCD shall be the square required.



Proof. In the two triangles BEA, DEA,

because $\begin{cases} \text{BE} = \text{DE, being radii,} \\ \text{and AE is common,} \\ \text{and the right } \angle\text{BEA} = \text{the right } \angle\text{DEA,} \end{cases}$

∴ the base BA = the base DA. [I. 4.

Similarly, BC, DC each = BA, or DA;

∴ the figure ABCD is equilateral.

Also, BD being a diameter of the circle ABCD,

BAD is a semicircle; [*Constr.*

∴ the ∠BAD is a right ∠. [III. 31.

∴ the figure ABCD is equilateral, and has one angle a right angle;

∴ it is a square.

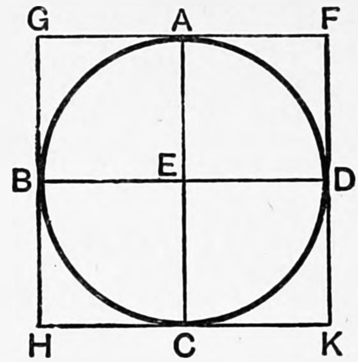
Wherefore *a square has been inscribed in the given circle.*

[Q.E.F.

PROPOSITION 7. PROBLEM.

To describe a square about a given circle.

Let ABCD be the given circle:
it is required to describe a square about it.



Construction. Find the centre E, and draw two diameters AC, BD at right angles to one another; [III. 1, I. 11.
and through A, B, C, D draw FG, GH, HK, KF perpendicular to EA, EB, EC, ED.

The figure GHKF shall be the square required.

Proof. (1) Because FG, GH, HK, KF are drawn at right angles to the radii EA, EB, EC, ED at their extremities,

therefore they touch the circle, [III. 16, *Cor.* 1.

and the circle is inscribed in the figure GHKF.

Also because AEB and EBG are both right angles, [*Constr.*

∴ GH is parallel to AC. [I. 28.

Similarly, AC is parallel to FK,

and GF, HK are each parallel to BD;

∴ the figures GK, GC, CF, FB, BK are parallelograms;

∴ GH and FK each = AC,

and GF and HK each = BD. [I. 34.

But AC = BD, both being diameters;

∴ GH, HK, KF, FG are all equal, and FGHK is equilateral.

(2) Again, since AEBG is a parallelogram, and AEB a right ∠,

∴ the opposite ∠AGB is also a right ∠. [I. 34.

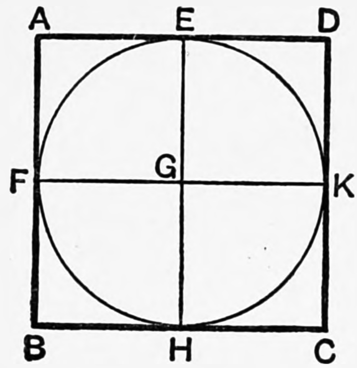
∴ the figure FGHK is equilateral, and has one of its angles a right angle; therefore it is a square.

Wherefore *a square has been described about the given circle.* [Q.E.F.

PROPOSITION 8. PROBLEM.

To inscribe a circle in a given square.

Let ABCD be the given square:
it is required to inscribe a circle in ABCD.



Construction. Bisect each of the sides AB, AD at the points F, E; [I. 10.
through E draw EH parallel to AB or DC, and through F draw FK parallel to AD or BC. [I. 31.

Proof. Let EH and FK meet in G.
AB and AD are equal, being sides of a square;
∴ their halves, AF and AE, are equal; [*Axiom* 7.
therefore, since AEGF is a parallelogram by construction, the opposite sides GF, GE are equal. [I. 34.

Similarly, it may be shewn that GK = GE, and GH = GF.
∴ GE, GF, GH, GK are all equal, and the circle described with centre G, and radius equal to any one of them, will pass through the extremities of the other three;
and it will touch AB, BC, CD, DA, because these are straight lines drawn through E, F, H, K perpendicular to the radii. [III. 16, *Cor.* 1.

Wherefore *a circle has been inscribed in the given square.* [Q.E.F.

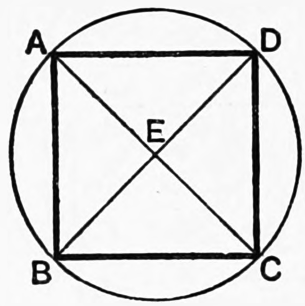
EXERCISE.

Prove the following alternative construction: *Draw the diagonals* AC *and* BD, *and let them meet in* O; *draw* OP, OQ, OR, OS *perpendicular to the sides; the required circle has its centre at* O *and its radius equal to either of the four* OP, OQ, OR, OS.

PROPOSITION 9. PROBLEM.

To describe a circle about a given square.

Let ABCD be the given square:
it is required to describe a circle about ABCD.



Construction. Join AC, BD, cutting one another at E.

Proof. In the triangles BAC, DAC,

because $\begin{cases} AB = AD, \\ \text{and } AC \text{ is common,} \\ \text{and the base } BC = \text{the base } DC; \end{cases}$

∴ the $\angle BAC$ = the $\angle DAC$, [I. 8.

that is, the $\angle BAD$ is bisected by AC.

Similarly, the other angles of the square are bisected by BD or AC.

Then, because the angles DAB, ABC are equal,
∴ their halves, the angles EAB, EBA, are equal,
and therefore the sides EB and EA are equal. [I. 6.

Similarly, it may be shewn that EC = EB, and ED = EA.
∴ EA, EB, EC, ED are all equal, and the circle described with centre E, and radius equal to any one of them, will pass through the ends of the other three, and will be described about the square ABCD.

Wherefore *a circle has been described about the given square.*

[Q.E.F.

EXERCISES.

1. Describe a circle about a given rectangle.

****2.** The square inscribed in a circle is double of the square on the radius.

****3.** The square circumscribed about a circle is double of the square inscribed in the same circle.

****4.** Shew that no rectangle except a square can be described about a circle.

5. Describe a square about a given rectangle.

[Let ABCD be the rectangle; at A draw a straight line outside the square making an $\angle$ equal to half a right $\angle$ with AD; similarly at B, C, D; the figure obtained is the required square.]

6. Inscribe a regular octagon in a given circle.

7. The area of a regular octagon inscribed in a circle is equal to the rectangle contained by the sides of the squares inscribed in and circumscribed about the circle.

[Let ABCD be the inscribed square and E, F, G, H the middle points of the arcs AB, BC, CD, DA. Let the tangents at A, B meet in K and AB meet OK in L. The rect. contained by the sides of the inscribed and circumscribed squares $=$ rect. by 2 AK and AB $=$ four times rect. AK, AL $=$ four times rect. AL, OB $=$ four times rect. AL, OE $=$ eight times $\triangle$ AOE (I. 41) $=$ area of octagon AEBFCGDK.]

8. If from any point in the circumference of a given circle straight lines be drawn to the four angular points of an inscribed square, the sum of the squares on the four straight lines is double the square on the diameter. [Use Ex. 1, p. 109.]

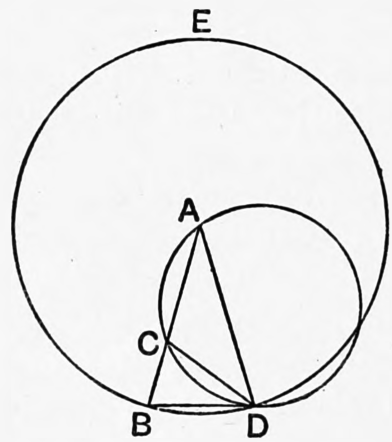
Proposition 10. Problem

To describe an isosceles triangle, having each of the angles at the base double of the third angle.

Construction. Take any straight line AB, and divide it at C, so that the rect. AB, BC = the square on AC; [II. 11.
with centre A and radius AB describe the circle BDE, and in it draw the chord BD equal to AC; [IV. 1.
join DA.

The triangle ABD shall be such as is required.

Join DC; and about the triangle ACD describe the circle ACD. [IV. 5.



Proof. Because the rect. AB, BC = the square on AC, [*Const.*
that is, = the square on BD, [*Construction.*
∴ BD touches the circle ACD. [III. 37.

Also, DC is drawn from the point of contact D;
∴ the ∠BDC = the ∠DAC in the alternate segment. [III. 32.

To each of these add the ∠CDA;

∴ the whole angle BDA = the two angles CDA, DAC. [*Ax.* 2.

But the exterior ∠BCD = the angles CDA, DAC; [I. 32.

∴ the ∠BDA = the ∠BCD. [*Axiom* 1.

But the ∠BDA = the ∠DBA, since AD = AB; [I. 5.
∴ the ∠DBA also = the ∠BCD; [*Axiom* 1.

∴ DC = DB; [I. 6.

but DB was made equal to CA; ∴ CA = CD, [*Axiom* 1.

and ∴ the ∠CAD = the ∠CDA. [I. 5.

But the angle BCD = the sum of the angles CAD, CDA; [I. 32.
∴ the ∠BCD is double of the ∠CAD.
And the angle BCD has been shewn to be equal to each of the angles BDA, DBA;
∴ each of the angles BDA, DBA is double of the ∠BAD.

Wherefore *an isosceles triangle has been described, having each of the angles at the base double of the third angle.* [Q.E.F.

EXERCISES.

****1.** Prove that the angle BAD is one-fifth part of two right angles.

****2.** Divide a right angle into 5 equal parts.

3. Divide a circle into two parts so that the angle in one segment may be four times that in the other.

4. Shew that the angle ACD in the figure of IV. 10 is equal to three times the angle at the vertex of the triangle.

5. Shew that the smaller of the two circles employed in the figure of IV. 10 is equal to the circle described round the required triangle.

[If two △ˢ have equal bases and equal vertical angles then, as in the converse of III. 21, their circum-circles are equal. Also, in the above figure, we have BD = AC and ∠BAD = ∠ADC; ∴ etc.]

6. Shew that in the figure of IV. 10 there are two triangles which possess the required property. Shew that there is also an isosceles triangle whose equal angles are each one-third part of the third angle.

[These two △ˢ are BCD and ACD.]

In the figure of IV. 10, if the two circles meet again in F, prove that

****7.** AC is the side of a regular decagon inscribed in the larger circle.

****8.** BD ,, ,, pentagon ,, smaller circle.

9. DF = BD.

[∠AFD = 2 rt. ∠ˢ − ∠ACD = ∠BCD = ∠ABD, etc.]

10. BF is the side of a regular pentagon inscribed in the larger circle.

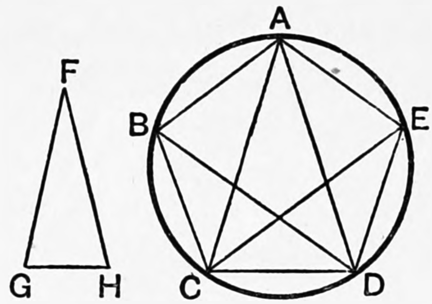
[∠FAD = ∠BAD; ∴ ∠FAB = 2∠BAD = one fifth of four rt. ∠ˢ; ∴ etc.]

11. If AF meet BD in K, then CDKF is a parallelogram.

Proposition 11. Problem.

To inscribe a regular pentagon in a given circle.

Let ABCDE be the given circle :
it is required to inscribe a regular pentagon in the circle ABCDE.



Construction. Describe an isosceles triangle FGH, having each of the angles at G, H double of the angle at F ; [IV. 10.
in the circle ABCDE inscribe the triangle ACD equiangular to the triangle FGH, [IV. 2.
so that each of the angles ACD, ADC is double of the angle CAD ;
bisect the angles ACD, ADC by the straight lines CE, DB, [I. 9.
and join AB, BC, AE, ED ;
then ABCDE shall be the pentagon required.

Proof. (1) Because each of the angles ACD, ADC is double of the angle CAD,
and that they are bisected by the straight lines CE, DB ;
∴ the five angles ADB, BDC, CAD, DCE, ECA are equal.
But equal angles stand on equal arcs ;
∴ the five arcs AB, BC, CD, DE, EA are equal. [III. 26.
Also equal arcs are subtended by equal chords ; [III. 29.
∴ the five chords AB, BC, CD, DE, EA are equal ;
and therefore the pentagon ABCDE is equilateral.

(2) It is also equiangular.
For the arc AB = the arc DE ; to each add the arc BCD ;
∴ the whole arc ABCD = the whole arc BCDE. [*Axiom* 2.

Also the $\angle$AED stands on the arc ABCD, and the $\angle$BAE on the arc BCDE.

$\therefore$ the $\angle AED =$ the $\angle BAE$. [III. 27.

Similarly, each of the angles ABC, BCD, CDE is equal to the angle AED or BAE;

$\therefore$ the pentagon ABCDE is equiangular.

Also it has been shewn to be equilateral.

Wherefore *a regular pentagon has been inscribed in the given circle.* [Q.E.F.

EXERCISES.

**1. What is the magnitude of the angle of a regular pentagon?

2. If the alternate sides of a regular pentagon be produced to meet, the five points of intersection form another regular pentagon.

ABCDE is a regular pentagon; prove that

**3. Any angle of it is trisected by the straight lines joining it to the opposite angular points.

4. CD and AE meet at an angle equal to CAD.

[If CD, AE meet in X, the $\angle XDE = \angle CAE$ (III. 22) $= 2 \angle CAD = \angle ACD$ and so $\angle XED = \angle ADC$; $\therefore$ $\angle EXC = \angle CAD$.]

**5. The diagonals AC and AD are parallel to the sides ED and BC respectively.

**6. All the diagonals intersect so as to form another regular pentagon.

7. If AC and BE meet in F, then AB, CF, and EF are equal, and hence that any two diagonals divide one another in medial section.

[$\angle FEC = \angle FCE = \angle$ subtended by a side; $\therefore$ FE = FC; $\angle CFB = \angle FBA + \angle FAB =$ twice $\angle$ subtended by a side $= \angle CBE$; $\therefore$ CF = CB = AB. Also $\triangle$ACE is a $\triangle$ equiangular with $\triangle$ABD in IV. 10 and since $\angle AEB = \angle BEC$, EF bisects $\angle$E at its base; $\therefore$ as in IV. 10, rect. CA . AF $= CF^2$.]

8. CFED is a rhombus.

9. AB is a tangent to the circum-circle of the triangle BFC.

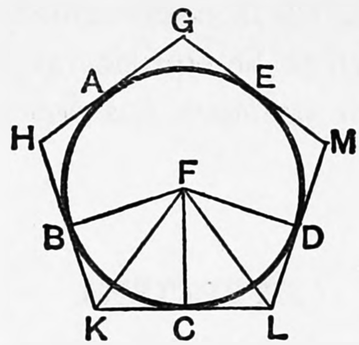
10. AE and BC are tangents to the circum-circle of the triangle CFE.

11. ABCDE is a regular pentagon inscribed in a circle; the arcs AB, BC, CD, DE, and EA are bisected, and lines drawn parallel to the sides of the inscribed pentagon. Prove that these lines form a circumscribing pentagon.

Proposition 12. Problem.

To describe a regular pentagon about a given circle.

Let ABCDE be the given circle:
it is required to describe a regular pentagon about it.



Construction. Let the angular points of a regular pentagon, inscribed in the circle, be A, B, C, D, E, so that the arcs AB, BC, CD, DE, EA are equal;
and through the points A, B, C, D, E draw GH, HK, KL, LM, MG touching the circle. [III. 17.
The figure GHKLM shall be the pentagon required.

Proof. (1) In the triangles FBK, FCK,

because $\begin{cases} \text{FB} = \text{FC, both being radii,} \\ \text{and FK is common,} \\ \text{and the base BK} = \text{the base KC, both being tangents.} \end{cases}$ [III. 17, *Cor.* 2.

∴ the ∠BFK = the ∠CFK, [I. 8.
and the ∠BKF = the ∠CKF; [I. 4.
∴ the ∠BFC = twice the ∠CFK,
and the ∠BKC = twice the ∠CKF.
Similarly, the ∠CFD = twice the ∠CFL,
and the ∠CLD = twice the ∠CLF.

And because the arc BC = the arc CD,
the ∠BFC = the ∠CFD; [III. 27.
and the ∠BFC is double of the ∠CFK, and the ∠CFD is double of the ∠CFL;
∴ the ∠CFK = the ∠CFL. [*Axiom* 7.

Then, in the triangles FCK, FCL,

because $\begin{cases} \text{the } \angle CFK = \text{the } \angle CFL, & \text{[}\textit{Proved.}\\ \text{and the right } \angle FCK = \text{the right } \angle FCL, \\ \text{and the side FC is common;} \end{cases}$

$\therefore$ CK = CL, and the $\angle$FKC = the $\angle$FLC. [I. 26.

Also because CK is equal to CL, LK is double of CK.

Similarly, it may be shewn that HK is double of BK.

And because BK is equal to CK, as was shewn, and that HK is double of BK, and LK double of CK;

$\therefore$ HK = LK. [*Axiom* 6.

Similarly, it may be shewn that any two consecutive sides of the pentagon are equal, and it is therefore equilateral.

(2) It is also equiangular.

For since the $\angle$FKC = the $\angle$FLC, and that the $\angle$HKL is double of the $\angle$FKC, and the $\angle$KLM double of the $\angle$FLC, as was shewn;

$\therefore$ the $\angle$HKL = the $\angle$KLM. [*Axiom* 6.

In the same manner it may be shewn that any two consecutive angles of the pentagon are equal, and it is therefore equiangular.

Also it has been shewn to be equilateral.

Wherefore *a regular pentagon has been described about the given circle.* [Q.E.F.

EXERCISES.

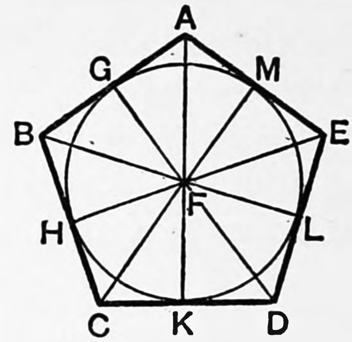
**1. Prove the following alternative construction: *In the circle inscribe, by* IV. 11, *a regular polygon* ABCDE; *draw the tangents to the circle at* A, B, C, D, E, *and let them meet in* P, Q, R, S, T; *then* PQRST *is the required figure.*

**2. The bisectors of all the angles of a regular polygon meet in a point.

Proposition 13. Problem.

To inscribe a circle in a given regular pentagon.

Let ABCDE be the given regular pentagon :
it is required to inscribe a circle in it.



Construction. Bisect the angles BCD, CDE by CF, DF; [I. 9.

and from the point F, at which they meet, draw the straight lines FB, FA, FE.

Draw FG, FH, FK, FL, FM perpendiculars to AB, BC, CD, DE, EA. [I. 12.

Proof. In the triangles BCF, DCF,

because { BC = CD, [*Hypothesis.*
and CF is common,
and the ∠BCF = the ∠DCF ; [*Construction.*

∴ the base BF = the base DF,
and the ∠CBF = the ∠CDF, [I. 4.

∴ twice the ∠CBF = twice the ∠CDF.
that is, = the ∠CDE, which = the ∠CBA.

that is, the ∠ABC is bisected by BF.

In the same manner it may be shewn that the angles BAE, AED are bisected by AF, EF.

Again, in the triangles FHC, FKC,

because { the ∠FCH = the ∠FCK,
and the right ∠FHC = the right ∠FKC,
and the side FC is common ;

∴ FH = FK. [I. 26.

Similarly, it may be shewn that FL, FM, FG are each equal to FH or FK ;

∴ FG, FH, FK, FL, FM are all equal, and the circle described, with centre F and radius equal to any one of them, will pass through the extremities of the other four ;

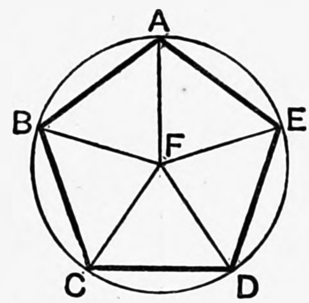
and it will touch AB, BC, CD, DE, EA, because the angles at G, H, K, L, M are right angles. [III. 16.

Wherefore *a circle* GHKLM *has been inscribed in the given regular pentagon* ABCDE. [Q.E.F.

PROPOSITION 14. PROBLEM.

To describe a circle about a given regular pentagon.

Let ABCDE be the given regular pentagon:
it is required to describe a circle about it.



Construction. Bisect the angles BCD, CDE by the straight lines CF, DF, which meet in F. [I. 9.
Join FB, FA, FE.

Proof. It may be shewn, as in the preceding proposition, that the angles CBA, BAE, AED are bisected by the straight lines BF, AF, EF.

Also because the $\angle$BCD = the $\angle$CDE,
and that the $\angle$FCD is half of the $\angle$BCD,
and the $\angle$FDC is half of the $\angle$CDE,
$\therefore$ the $\angle$FCD = the $\angle$FDC; [*Axiom* 7.
$\therefore$ FC = FD. [I. 6.

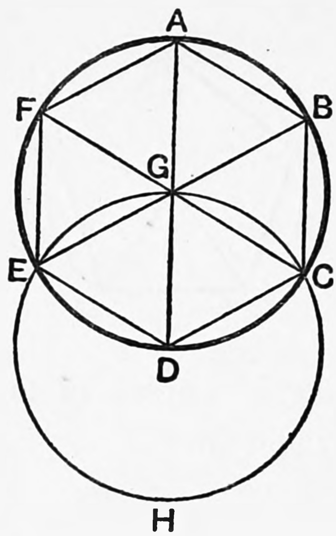
In the same manner it may be shewn that FB, FA, FE are each equal to FC or FD;
therefore FA, FB, FC, FD, FE are all equal, and the circle described, with centre F and radius equal to any one of them, will pass through the extremities of the other four, and will be described about the pentagon ABCDE.

Wherefore *a circle has been described about the given regular pentagon.* [Q.E.F.

PROPOSITION 15. PROBLEM.

To inscribe a regular hexagon in a given circle.

Let ABCDEF be the given circle: it is required to inscribe a regular hexagon in it.



Construction. Find the centre G of the circle, [III. 1.
and draw the diameter AGD;
with centre D and radius DG describe the circle EGCH,
join EG, CG, and produce them to meet the circumference of the circle ABCDEF in B and F;
join AB, BC, CD, DE, EF, FA.
Then ABCDEF is the hexagon required.

Proof. (1) Because

GE = GD, being radii of the circle ABCDEF,

and DE = DG, being radii of the circle EGCH,

∴ the triangle EGD is equilateral; [*Axiom* 1.

∴ the angles EGD, GDE, DEG are all equal. [I. 5, *Corollary.*

∴ the angle EGD = one-third of two right angles. [I. 32.

Similarly, it may be shewn that the angle DGC = one-third of two right angles.

∴ the angle EGC = two-thirds of two right angles;

but the angles EGC, CGB together = two right angles; [I. 13.

∴ the angle CGB = one-third of two right angles;
∴ the angles EGD, DGC, CGB are equal.

Also to these are equal the vertical opposite angles BGA, AGF, FGE; [I. 15.
∴ the six angles EGD, DGC, CGB, BGA, AGF, FGE are all equal;
∴ the six arcs AB, BC, CD, DE, EF, FA are all equal. [III. 26.
∴ the six chords AB, BC, CD, DE, EF, FA are all equal, and the hexagon is equilateral. [III. 29.

(2) It is also equiangular.

For the arc AF = the arc ED; to each add the arc ABCD;
∴ the whole arc FABCD = the whole arc ABCDE;
therefore the angles FED, AFE, which stand on these equal arcs, are equal. [III. 27.

Similarly, it may be shewn that any other two angles of the hexagon ABCDEF are equal;
∴ the hexagon is equiangular, and it has been shewn to be equilateral, and it is inscribed in the circle ABCDEF.

Wherefore *a regular hexagon has been inscribed in the given circle.* [Q.E.F.

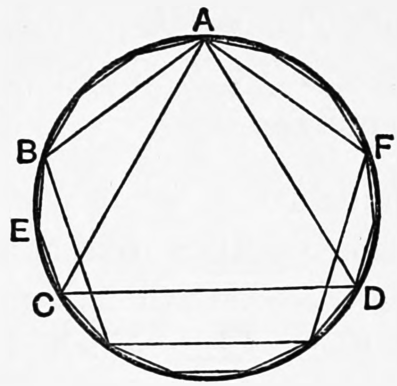
Corollaries. (1) Since EDG is an equilateral triangle, therefore DE = DG,
that is, the side of the hexagon = the radius of the circle.

(2) If through the points A, B, C, D, E, F there be drawn tangents to the circle, a regular hexagon will be described about the circle, as may be shewn from what was said of the pentagon; and a circle may be inscribed in a given regular hexagon, and circumscribed about it, by a method like that used for the pentagon.

PROPOSITION 16. PROBLEM.

To inscribe a regular quindecagon in a given circle.

Let ABCD be the given circle:
it is required to inscribe a regular quindecagon in it.



Let AC be the side of an equilateral △ inscribed in the circle, [IV. 2.
and AB the side of a regular pentagon inscribed in it. [IV. 11.

Then, of such equal parts, as the whole circumference ABCDF contains fifteen,
the arc ABC, which is the third part of the whole, contains five,
and the arc AB, which is the fifth part, contains three,
and therefore their difference, the arc BC, contains two.
Bisect the arc BC at E; [III. 30.
therefore each of the arcs BE, EC is the fifteenth part of the whole circumference.
∴ if BE, EC be drawn, and chords equal to them be placed round in the whole circle, [IV. 1.
a regular quindecagon will be inscribed in it. [Q.E.F.

Corollary. As in the case of the pentagon, if through the points of division made by inscribing the quindecagon tangents be drawn to the circle, a regular quindecagon will be described about it; and also, as for the pentagon, a circle may be inscribed in a given regular quindecagon, and circumscribed about it.

EXERCISES.

1. AB, BC are sides of a regular hexagon, and therefore AC is the side of an equilateral triangle inscribed in the same circle. Prove that the square on AC is three times the square on AB.

2. In the figure of IV. 15, prove that A, C, E are the angular points of an equilateral triangle.

****3.** The area of a regular hexagon is twice that of an equilateral triangle inscribed in the same circle.

[In the figure of IV. 15, AGBC is a $\|^{\text{gm}}$, and $\therefore$ = twice $\triangle$ AGC. But the hexagon = three times AGCB and $\triangle$ AEC = three times AGC, etc.]

4. Construct on a given straight line as one side a regular—

(1) pentagon ; (2) hexagon ; (3) octagon.

****5.** Inscribe a figure of 30 sides in a given circle.

****6.** Any equilateral figure which is inscribed in a circle is also equiangular.

7. If the alternate sides of a regular polygon be produced to meet, the points of intersection form another regular polygon of the same number of sides.

****8.** Prove that the centre of the inscribed circle and of the circumscribing circle of any regular polygon is found by bisecting any two consecutive angles.

****9.** If we have any regular polygon in a circle and draw tangents to the circle at its vertices, the latter form a regular circumscribed polygon having the same number of the sides as the first.

10. Verify that, by the constructions of this book, we can inscribe in a circle polygons of respectively 2×2^n, 3×2^n, 5×2^n, and 15×2^n sides, where n is any positive integer.

[See also Notes, page 328.]

NOTES ON EUCLID'S ELEMENTS.

THE article "Eucleides," in Dr. Smith's *Dictionary of Greek and Roman Biography*, was written by Professor De Morgan; it contains an account of the works of Euclid, and of the various editions of them which have been published. To that article we refer the student who desires full information on these subjects. Perhaps the only work of importance relating to Euclid which has been published since the date of that article is a work on the *Porisms of Euclid*, by Chasles. Paris, 1860.

Euclid appears to have lived in the time of the first Ptolemy, B.C. 323–283, and to have been the founder of the Alexandrian mathematical school. The work on Geometry known as *The Elements of Euclid* consists of thirteen books; two other books have sometimes been added, of which it is supposed that Hypsicles was the author. Besides the *Elements*, Euclid was the author of other works, some of which have been preserved and some lost.

We will now mention the three editions which are the most valuable for those who wish to read the *Elements of Euclid* in the original Greek.

(1) The Oxford edition in folio, published in 1703, by David Gregory, under the title *Εὐκλείδου τὰ σωζόμενα*. "As an edition of the whole of Euclid's works, this stands alone, there being no other in Greek."—*De Morgan.*

(2) *Euclidis Elementorum Libri sex priores*, edidit Joannes Gulielmus Camerer. This edition was published at Berlin in two volumes octavo, the first volume in 1824 and the second in 1825. It contains the first six books of the *Elements* in Greek, with a Latin translation, and very good notes, which form a mathematical commentary on the subject.

(3) *Euclidis Elementa ex optimis libris in usum tironum Græce, edita ab Ernesto Ferdinando August.* This edition was published at Berlin in two volumes octavo, the first volume in 1826 and the second in 1829. It contains the thirteen books of the *Elements* in Greek, with a collection of various readings. A third volume, which was to have contained the remaining works of Euclid, never appeared. "To the scholar who wants one edition of the *Elements* we should decidedly recommend this, as bringing together all that has been done for the text of Euclid's greatest work."—*De Morgan.*

An edition, in five volumes, of the whole of Euclid's works in the original has been issued by Teubner, the well-known German publisher, as one of his series of compact editions of Greek and Latin authors.

Robert Simson's edition of the *Elements of Euclid*, which we have in substance adopted in the present work, differs considerably from the original. The English reader may ascertain the contents of the original by consulting the work entitled *The Elements of Euclid with dissertations*, by James Williamson. This work consists of two volumes quarto; the first volume was published at Oxford in 1781, and the second at London in 1788. Williamson gives a close translation of the thirteen books of the *Elements* into English, and he indicates by the use of italics the words which are not in the original, but which are required by our language.

For the history of Geometry the student is referred to Montucla's *Histoire des Mathématiques*, and to Chasles's *Aperçu historique sur l'origine et le devéloppement des Méthodes en Géométrie.*

BOOK I.

Definitions. The first seven definitions have given rise to considerable discussion, on which however we do not propose to enter. Such a discussion would consist mainly of two subjects, both of which are unsuitable to an elementary work, namely, an examination of the origin and nature of some of our elementary ideas, and a comparison of the original text of Euclid with the substitutions for it proposed by Simson and other editors. For the former subject the student may hereafter consult Whewell's *History of Scientific Ideas* and Mill's *Logic*, and for the latter the notes in Camerer's edition of the *Elements of Euclid.*

We will only observe that the ideas which correspond to the words *point*, *line*, and *surface*, do not admit of such definitions as will really supply the ideas to a person who is destitute of them. The so-called definitions may be regarded as cautions or restrictions.

Thus a *point* is not to be supposed to have any *size*, but only *position*;
a *line* is not to be supposed to have any *breadth* or *thickness*, but only *length*;
a *surface* is not to be supposed to have any *thickness*, but only *length* and *breadth*.

The eighth definition seems intended to include the cases in which an angle is formed by the meeting of two *curved* lines, or of a *straight* line and a *curved* line; this definition however is of no importance, as the only angles ever considered are such as are formed by straight lines.

Some writers object to such definitions as those of an equilateral triangle, or of a square, in which the existence of the object defined is *assumed* when it ought to be *demonstrated.* They would present them in such a form as the following: if there be a triangle having three equal sides, let it be called an equilateral triangle.

Moreover, some of the definitions are introduced prematurely. Thus, for example, take the definitions of a right-angled triangle and an obtuse-angled triangle; it is not shewn until I. 17 that a triangle cannot have both a right angle and an obtuse angle, and so cannot be at the same time right-angled and obtuse-angled. And before Axiom 11 has been given, it is conceivable that the same angle may be greater than one right angle, and less than another right angle, that is, obtuse and acute at the same time.

On the *method of superposition* we may refer to papers by Professor Kelland in the *Transactions of the Royal Society of Edinburgh*, Vols. XXI. and XXIII.

The first book is chiefly devoted to the properties of triangles and parallelograms.

We may observe that Euclid himself does not distinguish between problems and theorems except by using at the end of the investigation phrases which correspond to Q.E.F. and Q.E.D. respectively.

I. 2. This problem admits of *eight* cases in its figure. For it will be found that the given point may be found with *either* end of the given straight line; there the equilateral triangle may be described on *either* side of the straight line which is drawn, and the sides of the equilateral triangle which are produced may be produced through *either* extremity. These various cases may be left for the exercise of the student, as they present no difficulty.

There will not however always be eight different straight lines obtained which solve the problem. For example, if the point A falls on BC produced, some of the solutions obtained coincide; this depends on the fact which follows from I. 32, that the angles of all equilateral triangles are equal.

I. 5. "Join FC." Custom seems to allow this singular expression as an abbreviation for "draw the straight line FC," or for "join F to C by the straight line FC."

It has been suggested to demonstrate I. 5 by *superposition.* Conceive the isosceles triangle ABC to be taken up, and then replaced so that AB falls on the old position of AC, and AC falls on the old position of AB. Thus, in the manner of I. 4, we can shew that the angle ABC is equal to the angle ACB.

I. 6. This proposition is not required by Euclid before he reaches II. 4; so that I. 6 might be removed from its present place and demonstrated hereafter in other ways if we please. For example, I. 6 might be placed after I. 18, and demonstrated thus. Let the angle ABC be equal to the angle ACB; then the side AB shall be equal to the side AC. For if not, one of them must be greater than the other; suppose AB greater than AC. Then the angle ACB is greater than the angle ABC, by I. 18. But this is impossible, because the angle ACB is equal to the angle ABC, by hypothesis. Or I. 6 might be placed after I. 26, and demonstrated thus. Bisect the angle BAC by a straight line meeting the base at D. Then the triangles ABD and ACD are equal in all respects, by I. 26.

I. 12. Here the straight line is said to be of *unlimited* length, in order that we may ensure that it shall meet the circle.

Euclid distinguishes between the terms *at right angles* and *perpendicular.* He uses the term *at right angles* when the straight line is drawn from a point *in* another, as in I. 11; and he uses the term *perpendicular* when the straight line is drawn from a point *without* another, as in I. 12. This distinction, however, is often disregarded by modern writers.

I. 20. "Proclus, in his *Commentary*, relates, that the Epicureans derided Prop. 20, as being manifest even to asses, and needing no demonstration; and his answer is, that though the truth of it be manifest to our senses, yet it is science which must give the reason why two sides of a triangle are greater than the third: but the right answer to this objection, against this and the 21st, and some other plain propositions, is, that the number of axioms ought not to be increased without necessity, as it must be if these propositions be not demonstrated."—*Simson.*

I. 21. Here it must be carefully observed that the two straight lines are to be drawn *from the ends of the side* of the triangle. If this condition be omitted the two straight lines will not necessarily be less than two sides of the triangle.

I. 22. "Some authors blame Euclid because he does not demonstrate that the two circles made use of in the construction of this problem must cut one another: but this is very plain from the determination he has given, namely, that any two of the straight lines A, B, C, must be greater than the third.

The condition that B and C are greater than A, ensures that the circle described from the centre G shall not fall entirely within the circle described from the centre F; the condition that A and B are

greater than C, ensures that the circle described from the centre F shall not fall entirely within the circle described from the centre G; the condition that A and C are greater than B, ensures that one of these circles shall not fall entirely without the other. Hence the circles must meet.

I. 26. It will appear after I. 32 that two triangles which have two angles of the one equal to two angles of the other, each to each, have also their third angles equal. Hence we are able to include the two cases of I. 26 in one enunciation thus; *if two triangles have all the angles of the one respectively equal to all the angles of the other, each to each, and have also a side of the one, opposite to any angle, equal to the side opposite to the equal angle in the other, the triangles shall be equal in all respects.*

Besides the cases of I. 4, 8, 26 there are two other cases which will naturally occur to a student to consider, namely,

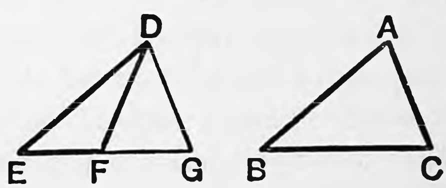
(1) when two triangles have the three angles of the one respectively equal to the three angles of the other;

(2) when two triangles have two sides of the one equal to two sides of the other, each to each, and an angle opposite to one side of one triangle equal to the angle opposite to the equal side of the other triangle.

In the first of these two cases the student will easily see, after reading I. 29, that the two triangles are not necessarily equal. In the second case also the triangles are not necessarily equal, as will be seen from a proposition which we shall now demonstrate.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles opposite to one pair of equal sides equal, the angle opposite the other pair of equal sides shall be either equal or supplementary.

Let ABC, DEF be two triangles having AB = DE, AC = DF, and the ∠ABC opposite AC = the ∠DEF opposite the equal side DF.



Let the △ABC be applied to the △DEF so that AB coincides with DE and the ∠ABC with the ∠DEF. Then the side BC falls on the side EF.

The point C then coincides with F, or falls in EF, or in EF produced.

If C coincides with F, the triangle ABC coincides with the triangle DEF, and they are equal in all respects.

If not, let C fall on the point G in EF, or EF produced.

The $\triangle^s$ABC, DEG then coincide and are equal in all respects, so that $\angle$DGE = $\angle$ACB, and the side DG = the side AC.

But AC = DF; $\therefore$ DG = DF; $\therefore$ $\angle$DGF = $\angle$DFG. [I. 6.

But the $\angle^s$DFG, DFE = two right angles.

$\therefore$ the $\angle^s$DGE, DFE = two right angles;

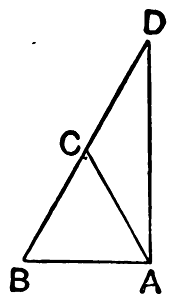
$\therefore$ the $\angle^s$ACB, DFE = two right angles;

that is, the $\angle^s$ACB, DFE are supplementary.

The student who wishes to examine the attempts that have been made to avoid the assumption made in Euclid's twelfth Axiom may consult Camerer's *Euclid*, Gergonne's *Annales de Mathématiques*, Volumes XV. and XVI., the work by Colonel Perronet Thompson entitled *Geometry without Axioms*, the article "Parallels" in the *English Cyclopædia*, a memoir by Professor Baden Powell in the second volume of the *Memoirs of the Ashmolean Society*, an article by M. Bouniakofsky in the *Bulletin de l'Académie Impériale*, Volume v., St. Petersburg, 1863, articles in the volumes of the *Philosophical Magazine* for 1856 and 1857, and a dissertation entitled *Sur un point de l'histoire de la Géométrie chez les Grecs*, par A. J. H. Vincent, Paris, 1857.

I. 32. If two triangles have two angles of the one equal to two angles of the other, each to each, they shall also have their third angles equal. This is a very important result, which is often required in the *Elements*. The student should notice how this result is established on Euclid's principles. By Axioms 11 and 2 one pair of right angles is equal to any other pair of right angles. Then, by I. 32, the three angles of one triangle are together equal to the three angles of any other triangle. Then, by Axiom 2, the sum of the two angles of one triangle is equal to the sum of the two equal angles of the other; and then, by Axiom 3, the third angles are equal.

After I. 32 we can draw a straight line at right angles to a given straight line from its extremity, without producing the given straight line.



Let AB be the given straight line. It is required to draw from A a straight line at right angles to AB.

On AB describe the equilateral triangle ABC. Produce BC to D, so that CD may be equal to CB. Join AD. Then AD shall be at right angles to AB. For the $\angle$CAD = the $\angle$CDA, and the $\angle$CAB = the $\angle$CBA, by I. 5. $\therefore$ the $\angle$BAD = the two angles ABD, BDA, by Axiom 2. Therefore the $\angle$BAD is a right $\angle$, by I. 32.

I. 35. The equality of the parallelograms in I. 35 is an equality of area, and not an identity of figure. Legendre proposed to use the word

equivalent to express the equality of area, and to restrict the word *equal* to the case in which magnitudes admit of superposition and coincidence. This distinction, however, has not been generally adopted, probably because there are few cases in which any ambiguity can arise; in such cases we may say especially, *equal in area*, to prevent misconception.

Cresswell, in his *Treatise of Geometry*, has given a demonstration of I. 35, which shews that the parallelograms may be divided into pairs of pieces admitting of superposition and coincidence; see also his Preface, page x.

I. 38. An important case of I. 38 is that in which the triangles are on equal bases and have a *common* vertex.

I. 40. We may demonstrate I. 40 without adopting the indirect method. Join BD, CD. The triangles DBC and DEF are equal, by I. 38; the triangles ABC and DEF are equal, by hypothesis; therefore the triangles DBC and ABC are equal, by the first Axiom. Therefore AD is parallel to BC, by I. 39.

THE SECOND BOOK.

The second book is devoted to the investigation of relations between the rectangles contained by straight lines divided into segments in various ways.

II. 2 and II. 3 are particular cases of II. 1.

II. 11. The student should notice that II. 11 gives a geometrical construction for the solution of a particular quadratic equation.

For if $AB = a$, and AH, the part to be found, $= x$, then $HB = a - x$, and the relation given, namely, rect. AB, BH = sq. on AH, says that

$$a(a-x) = x^2; \qquad \therefore\ x^2 + ax = a^2,$$

that is,

$$x^2 + ax + \frac{a^2}{4} = a^2 + \frac{a^2}{4} = \frac{5a^2}{4}.$$

Taking the square root, we have

$$x + \frac{a}{2} = \pm\frac{\sqrt{5}}{2}a, \text{ that is, } x = \frac{\sqrt{5}-1}{2}a \text{ or } -\frac{\sqrt{5}+1}{2}a.$$

The first value of x corresponds to the position of H found in the proposition. The second value corresponds to the position found in the first exercise following the proposition.

II. 12, II. 13. These are interesting in connection with I. 47; and, as the student may see hereafter, they are of great importance in Trigonometry; they are however not required in any of the parts of Euclid's *Elements* which are usually read. The converse of I. 47 is

proved in I. 48; and we can easily shew that converses of II. 12 and II. 13 are true.

Take the following, which is the converse of II. 12: *if the square described on one side of a triangle be greater than the sum of the squares described on the other two sides, the angle opposite to the first side is obtuse.*

For the angle cannot be a right angle, since the square described on the first side would then be equal to the sum of the squares described on the other two sides, by I. 47; and the angle cannot be acute, since the square described on the first side would then be less than the sum of the squares described on the other two sides, by II. 13; therefore the angle must be obtuse.

Similarly, we may demonstrate the following, which is the converse of II. 13: *if the square described on one side of a triangle be less than the sum of the squares described on the other two sides, the angle opposite to the first side is acute.*

II. 14. This is not required in any of the parts of Euclid's *Elements* which are usually read; it is included in VI. 22.

THE THIRD BOOK.

The third book of the *Elements* is devoted to properties of circles.

III. 1. In the construction, DC is said to be *produced* to E; this assumes that D is within the circle, which Euclid demonstrates in III. 2.

III. 3. This consists of two parts, each of which is the converse of the other; and the whole proposition is the converse of the corollary in III. 1.

The following proposition is analogous to III. 7 and III. 8.

If any point be taken on the circumference of a circle, of all the straight lines which can be drawn from it to the circumference, the greatest is that in which the centre is; and of any others, that which is nearer to the straight line which passes through the centre is always greater than one more remote; and from the same point there can be drawn to the circumference two straight lines, and only two, which are equal to one another, one on each side of the greatest line.

The first two parts of this proposition are contained in III. 15; all three parts might be demonstrated in the manner of III. 7.

III. 9. Euclid has given three demonstrations of III. 9, of which Simson has chosen the second. Euclid's other demonstration is as follows. Join D with the middle point of the straight line AB; then

it may be shewn that this straight line is at right angles to AB; and therefore the centre of the circle must lie in this straight line, by III. 1, Corollary. In the same manner it may be shewn that the centre of the circle must lie in the straight line which joins D with the middle point of the straight line BC. The centre of the circle must therefore be at D, because two straight lines cannot have more than one common point.

III. 10. Euclid has given two demonstrations of III. 10, of which Simson has chosen the second. Euclid's first demonstration resembles his first demonstration of III. 9. He shews that the centre of each circle is on the straight line which joins K with the middle point of the straight line BG, and also on the straight line which joins K with the middle point of the straight line BH; therefore K must be the centre of each circle.

The demonstration which Simson has chosen requires some additions to make it complete. For the point K might be supposed to fall *without* the circle DEF, or *on* its circumference, or *within* it; and of these three suppositions Euclid only considers the last. If the point K be supposed to fall *without* the circle DEF we obtain a contradiction of III. 8; which is absurd. If the point K be supposed to fall *on* the circumference of the circle DEF we obtain a contradiction of the proposition which we have enunciated at the end of the note on III. 7 and III. 8; which is absurd.

What is demonstrated in III. 10 is that the circumference of two circles cannot have more than *two* common points; there is nothing in the demonstration which assumes that the circles *cut* one another, but the enunciation refers to this case only because it is shewn in III. 13 that if two circles *touch* one another, their circumference cannot have more than *one* common point.

III. 11 may be deduced from III. 7. For GH is the least line that can be drawn from G to the circumference of the circle whose centre is F, by III. 7. Therefore GH is less than GA, that is, less than GD; which is absurd. Similarly, III. 12 may be deduced from III. 8.

III. 18. It does not appear that III. 18 adds anything to what we have already obtained in III. 16. For in III. 16 it is shewn that there is only one straight line which touches a given circle at a given point, and that the angle between this straight line and the radius drawn to the point of contact is a right angle.

III. 20. An important extension may be given to III. 20 by introducing angles greater than two right angles. For, in the first figure, suppose we draw the straight lines BF and CF. Then, the $\angle$BEA is double of the angle $\angle$BFA; and the $\angle$CEA is double of the $\angle$CFA;

∴ the sum of the angles BEA and CEA is double of the ∠BFC. The sum of the angles BEA and CEA is greater than two right angles; we will call the sum the *re-entrant* ∠BEC. Thus the re-entrant ∠BEC is double of the ∠BFC. (See note on I. 32.) If this extension be used some of the demonstrations in the third book may be abbreviated. Thus III. 21 may be demonstrated without making two cases; III. 22 will follow immediately from the fact that the sum of the angles at the centre is equal to four right angles; and III. 31 will follow immediately from III. 20.

III. 21. In III. 21 Euclid himself has given only the first case; the second case has been added by Simson and others. In either of the figures of III. 21 if a point be taken on the same side of BD as A, the angle contained by the straight lines which join this point to the extremities of BD is *greater* or *less* than the angle BAD, according as the point is *within* or *without* the circle BAD; this follows from I. 21.

III. 32. The converse of III. 32 is true and important; namely, *if a straight line meet a circle, and from the point of meeting a straight line be drawn cutting the circle, and the angle between the two straight lines be equal to the angle in the alternate segment of the circle, the straight line which meets the circle shall touch the circle.*

This may be demonstrated indirectly. For, if possible, suppose that the straight line which meets the circle does not touch it. Draw through the point of meeting a straight line to touch the circle. Then, by III. 32 and the hypothesis, it will follow that two different straight lines pass through the same point, and make the same angle, on the same side, with a third straight line which also passes through that point; but this is impossible.

III. 35, III. 36. The following proposition constitutes a large part of the demonstrations of III. 35 and III. 36. *If any point be taken in the base, or the base produced, of an isosceles triangle, the rectangle contained by the segments of the base is equal to the difference of the square on the straight line joining this point to the vertex and the square on the side of the triangle.*

This proposition is in fact demonstrated by Euclid, without using any property of the circle; if it were enunciated and demonstrated before III. 35 and III. 36 the demonstrations of these two propositions might be shortened and simplified.

THE FOURTH BOOK.

The fourth Book of the *Elements* consists entirely of problems. The first five propositions relate to triangles of any kind; the remaining

propositions relate to polygons which have all their sides equal and all their angles equal.

IV. 3. We can also describe a triangle equiangular to a given triangle, and such that one of its sides and the other two sides produced shall touch a given circle. For, in the figure of IV. 3, suppose AK produced to meet the circle again; and at the point of intersection draw a straight line touching the circle; this straight line, with parts of NB and NC, will form a triangle, which will be equiangular to the triangle MLN, and therefore equiangular to the triangle EDF; and one of the sides of this triangle, and the other two sides produced, will touch the given circle.

It was first demonstrated by Gauss in 1801, in his *Disquisitiones Arithmeticæ*, that it is possible to describe geometrically a regular polygon of 2^n+1 sides, provided 2^n+1 be a prime number; the demonstration is not of an elementary character. As an example, it follows that a regular polygon of seventeen sides can be described geometrically; this example is discussed in Catalan's *Théorèmes et Problèmes de Géométrie Elémentaire.*

APPENDIX.

THIS Appendix consists of a collection of important propositions which will be found useful, both as affording geometrical exercises, and as exhibiting results which are often required in mathematical investigations. The student will have no difficulty in drawing for himself the requisite figures in the cases where they are not given.

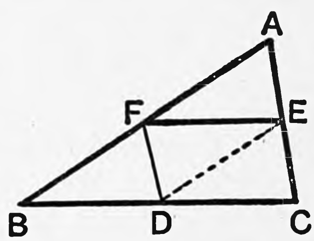
For convenience these propositions are arranged under the heading of the different Books.

THEOREMS AND EXAMPLES ON BOOK I.

1. *The straight line drawn through the middle point of the side of a triangle so as to be parallel to the base is equal to half of the base.*

Let F be the middle point of the side AB of the $\triangle$ ABC, and let FE drawn parallel to BC meet AC in E.

Then E *shall be the middle point of the side* AC.



Draw FD $\|^{l}$ to AC to meet BC in D.

$$\left.\begin{array}{l}\text{Since FE, BC are } \|^{l}, \ \therefore \angle AFE = \angle FBD. \\ \text{Since FD, AC are } \|^{l}, \ \therefore \angle BFD = \angle FAE. \\ \text{Also side BF} = \text{side FA.}\end{array}\right\} \text{[I. 29.}$$

$\therefore$ side AE = side FD, and base FE = base BD. [I. 26.

But, since FDCE is a $\|^{gm}$, by construction,

$\therefore$ FD = EC, and FE = DC.

$\therefore$ AE = EC, and BD = DC.

Conversely, it may be shewn by a *reductio ad absurdum* proof that if F, E be the middle points of AB, AC, then FE is parallel to BC.

Corollary. It follows at once that the area of the $\triangle$ AFE is one-quarter that of ABC.

For, if we join ED, we can shew as above that

$$\triangle \text{EDC} = \triangle \text{AFE} = \triangle \text{FBD}.$$

Also $\triangle$ EDC = $\triangle$ EDF, since EFDC is a $\|^{gm}$. [I. 34.

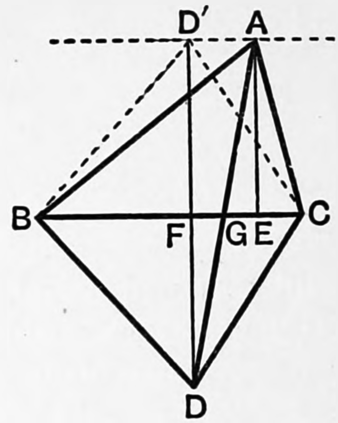
$\therefore$ the $\triangle^{s}$ AFE, FBD, EDC, FDE are all equal, and $\therefore$ each is one quarter of the whole $\triangle$ ABC.

2. *Two equal triangles are on the same base and on opposite sides of it; the straight line joining their vertices is bisected by the base.*

Let ABC, DBC be the equal triangles, and let AD meet BC in G.

Then shall AG = GD.

Draw AE, DF $\perp^r$ to BC and produce DF to D′ making FD′ = DF. Then from the $\triangle^s$ DCF, D′CF it is easily seen that D′C = CD; so D′B = BD.



∴ the $\triangle^s$ DBC, D′BC are equal in all respects.

∴ area $\triangle$ D′BC = area $\triangle$ DBC = area $\triangle$ ABC. [*Hypothesis.*

∴ AD′, BC are parallel, and ∴ AEFD′ is a $\|^{gm}$.

∴ AE = D′F = FD. [I. 34.

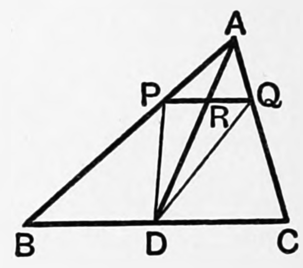
∴ in the $\triangle^s$ AGE, DGF we have $\angle$AGE = $\angle$DGF, rt. $\angle$AEG = rt. $\angle$DFG, and side AE = FD. ∴ AG = GD. [Q.E.D.

Corollary. It follows that all triangles with equal areas and equal bases have equal heights.

3. *Any median of a triangle bisects all lines parallel to the base to which it is drawn and intercepted by the sides.*

Let PQ, $\|^l$ to BC, meet the median AD through A in R.

Then shall PR = RQ.



Since BD = DC, ∴, by I. 38, $\triangle$ ABD = $\triangle$ ADC, and $\triangle$ PBD = $\triangle$ QDC.

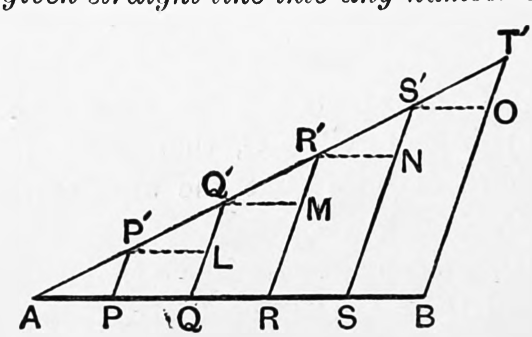
∴, by subtraction, $\triangle$ APD = $\triangle$ AQD.

i.e. the $\triangle^s$ APD, AQD on the same base are equal.

∴, by the previous theorem, PQ is bisected by AD.

Also PQ is any straight line $\|^l$ to the base. ∴ etc. [Q.E.D.

4. *To divide a given straight line into any number of given parts.*



Let AB be the given straight line, and let it be required to divide it into five given parts.

Through A draw a straight line AT′, making any angle with AB.

On it take any point P′ and cut off portions P′Q′, Q′R′, R′S′, S′T′ each equal to AP′.

Join T′B, and draw P′P, Q′Q, R′R, S′S all $\parallel^l$ to T′B to meet AB in P, Q, R, S respectively.

Then AB shall be divided at P, Q, R, S as required.

Draw P′L, Q′M, R′N, S′O $\parallel^l$ to AB to meet Q′Q, R′R, S′S, and T′B in L, M, N, O.

$$\text{Then } \angle Q'LP' = \angle LQP = \angle P'PA,$$

$$\text{and } \angle Q'P'L = \angle P'AP. \qquad \text{[I. 29.}$$

∴ since AP′ = P′Q′, the $\triangle^s$ APP′, P′LQ are equal in all respects.

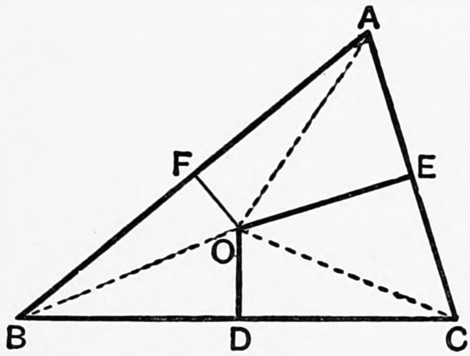
Similarly with the $\triangle^s$ Q′MR′, R′NS′, S′OT′.

$$\therefore\ AP = P'L = Q'M = R'N = S'O.$$

$$\therefore\ AP = PQ = QR = RS = SB. \qquad \text{[I. 34.}$$

The method is the same whatever be the number of parts into which AB is to be divided.

5. *The straight lines drawn at right angles to the sides of a triangle from the points of bisection of the sides meet in a point.*



Let ABC be a $\triangle$; bisect BC, CA, and AB at D, E, F respectively. Draw DO perp^r to BC and EO perp^r to CA, and let them meet in O. Join OA, OB, OC, OF.

In the $\triangle^s$ BDO, CDO we have BD = CD, DO common, and the rt. $\angle$BDO = the rt. $\angle$CDO; [I. 4.

$$\therefore\ BO = CO.$$

Similarly, from the $\triangle^s$ CEO, AEO we have CO = AO;

$$\therefore\ AO = BO.$$

Thus the $\triangle^s$ AFO, BFO have their sides respectively equal;

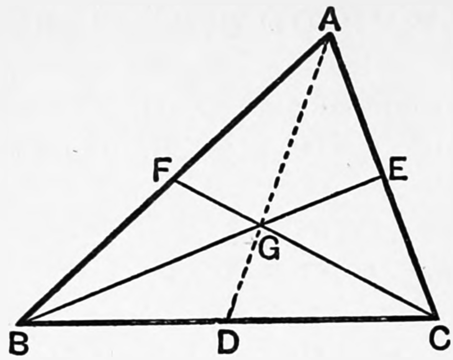
∴ the $\angle^s$ AFO, BFO are equal, and ∴ each is a rt. $\angle$; [I. 8 and *Def.* 10.

∴ FO bisects AB at rt. $\angle^s$; ∴ etc.

Corollary. Since AO = BO = CO, the point O is equidistant from the angular points of the $\triangle$.

When three straight lines, such as OA, OB, OC (or DO, EO, FO), meet in a point they are said to be **concurrent**.

6. *The straight lines drawn from the angles of a △ to bisect the opposite sides meet in a point.*



Let ABC be a △; D, E, F the middle points of its sides; let BE, CF meet in G.

Join AG, GD; *they shall be in the same straight line.*

The △BEA = △BEC, and △GEA = △GEC; [I. 38.

∴ by subtraction, △BGA = △BGC.

Similarly, △CGA = △CGB;

∴ △BGA = △CGA,

and △BGD = △CGD; [I. 38.

∴ the △s BGA, BGD together = △s CGA, CGD;

∴ the △s BGA, BGD together = half the △ABC;

∴ G must fall on the straight line AD,

that is, AG, GD are in one straight line. [I. 38.

7. *To prove* AG = 2 GD, BG = 2 GE, *and* CG = 2 GF.

Since the △s AGB, BGC, CGA are equal, each is one-third of △ABC.

∴ △AGB = one-third, that is two-sixths, of △ABC.

But △ADB = one-half, that is three-sixths, of △ABC;

∴ △BGD = one-sixth of △ABC;

∴ △AGB = twice △BGD.

If K be the middle point of AG we then have, by I. 38,

△ABK = △KBG = △GBD;

∴ AK = KG = GD; ∴ AG = 2 GD.

Similarly BG = 2 GE, and CG = 2 GF.

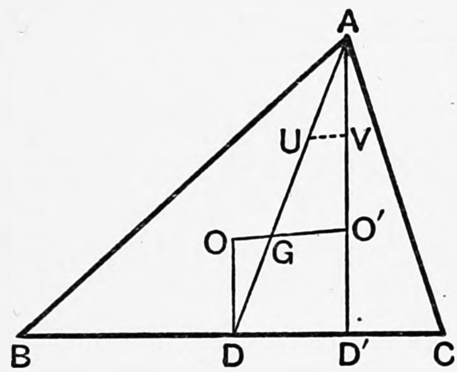
This point G is called the **Centroid** of the △ABC, and AD, BE, CF are the **Medians.**

8. *The perpendiculars drawn from the vertices of a triangle upon the opposite sides meet in a point.*

Let ABC be a △, O the point in which meet the perp^rs^ to the sides through the middle points, G the point in which meet the straight lines joining the vertices to opposite sides. [Arts. 5, 6.

Let D be the middle point of BC and AD′ the perp^r^ from A on BC.

Let OG meet AC in O′, and let U, V be the middle points of AG, AO′.



Then UV is parallel to, and one-half of, GO′. [Art. 1.

Because AD′, OD are parallel, both being perp^r^ to BC, [I. 28.

$$\therefore \angle ADO = \angle UAV.$$

Also, since UV, GO′ are parallel, $\therefore \angle AUV = \angle AGO' = \angle OGD$.

Also, $AU = \frac{1}{2} AG = GD$; [Art. 7.

$\therefore$ the △^s^ AUV, DGO are equal in all respects; [I. 4.

$$\therefore OG = UV = \tfrac{1}{2} GO'; \quad \therefore GO' = 2\,OG.$$

The point in which OG meets the perp^r^ from A on BC is thus a fixed point on OG produced; similarly the perp^rs^ BE, CF pass through this same fixed point O′;

Hence the three perpendiculars AD, BE, CF meet in a point O′ which lies on OG produced, and is such that $GO' = 2\,OG$.

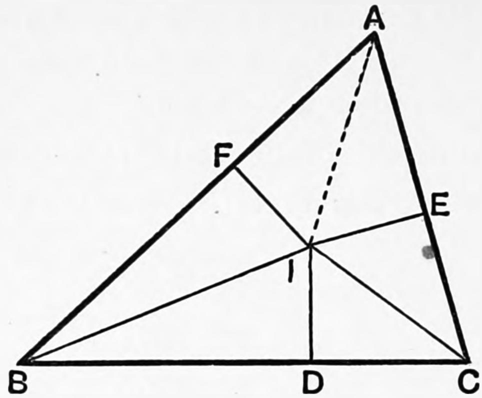
This point O′ is called the **Orthocentre**.

Corollary. Since the triangles AUV, DGO are equal,

$$\therefore OD = AV = \tfrac{1}{2} AO';$$

$$\therefore AO' = 2\,OD.$$

9. *The straight lines which bisect the angles of a triangle meet in a point.*



Let ABC be a triangle; bisect the angles at B and C by straight lines meeting at I; join AI:
then AI *shall bisect the angle at* A.

From I draw ID perpendicular to BC, IE perpendicular to CA, and IF perpendicular to AB.

In the $\triangle^s$ BIF, BID we have BF common, $\angle$DBI$=\angle$FBI, and the rt. $\angle$BDI$=$rt. $\angle$BFI; [*Construction.*

$\therefore$ IF$=$I . [I. 26.

Similarly from the $\triangle^s$ CID, CIE we have ID$=$IE;

$\therefore$ IE$=$IF.

Then since E, F are right $\angle^s$, we have $AF^2+FI^2=AI^2=AE^2+IE^2$;
$\therefore$ AF$=$AE, and the $\triangle^s$ AEI, AFI are equal in all respects;

$\therefore$ $\angle$IAE$=\angle$IAF.

Corollary. Since ID$=$IE$=$IF, the point I is such that its perpendicular distances from the three sides of the $\triangle$ are equal.

10. *Let two sides of a triangle be produced through the base; then the straight lines which bisect the two exterior angles thus formed, and the straight line which bisects the vertical angle of the triangle, meet at the same point.*

This may be shewn in the same manner as the last theorem.

11. *If two straight lines bisecting two angles of a triangle and terminated at the opposite sides be equal, the bisected angles shall be equal.*

Taking the figure of Art. 6, let CF$=$BE. Then, by Art. 7, GB$=$GC. Hence the $\triangle^s$ GDB, GDC are equal in all respects,

and $\therefore$ $\angle^s$ GBD, GDC are right angles.

Hence from the triangles ADB, ADC we have

AB$=$AC, and $\therefore$ $\angle$ABC$=\angle$ACB. [I. 5.

EXERCISES.

1. The sides AB, AC of a triangle ABC are bisected at D, E; BE and CD are drawn and produced to F and G so that $EF = BE$ and $DG = DC$; prove that AF and AG are in the same straight line.

[Use Art. 1.]

2. The middle points of the sides BC, CA, and AB of a triangle are D, E, F; FG is drawn parallel to BE to meet DE in G. Prove that the sides of the triangle CFG are equal to the medians of the triangle ABC, and hence that the sum of any two medians of a triangle is always greater than the third.

3. Prove that the sum of the medians of a triangle is less than the perimeter, but greater than three quarters of the perimeter, of the triangle.

[$GB + GC > BC$, *i.e.* $\frac{2}{3}BE + \frac{2}{3}CF > BC$, etc.
Also $AD < AE + ED$, *i.e.* $< \frac{1}{2}AC + \frac{1}{2}AB$, etc. (Fig. Art. 6)].

4. An angle of a triangle is right, acute, or obtuse according as the median drawn through it is $=$, $>$, or $<$ the side it bisects.

5. If one side of a triangle be longer than another, the corresponding median is shorter.

[Apply I. 25 to the $\triangle^s$ ADB, ADC (Fig. Art. 6), and then I. 24 to the $\triangle^s$ GDB, GDC.]

6. Of the two angles formed by a median with the two adjacent sides, that made with the shorter side is the greater.

7. If G be the centroid of a triangle ABC the triangle BGC is equal in area to the quadrilateral AEGF. [Fig. Art. 6.]

ON GEOMETRICAL ANALYSIS.

12. The substantives *analysis* and *synthesis*, and the corresponding adjectives *analytical* and *synthetical*, are of frequent occurrence in mathematics. In general *analysis* means decomposition, or the separating a whole into its parts, and *synthesis* means composition, or making a whole out of its parts. In Geometry, however, these words are used in a more special sense. In **synthesis** we begin with results already established, and end with some new result; thus, by the aid of theorems already demonstrated, and problems already solved, we demonstrate some new theorem, or solve some new problem. In **analysis** we begin with assuming the truth of some theorem or the solution of some problem, and we deduce from the assumption consequences which we can compare with results already established, and thus test the validity of our assumption.

13. The propositions in Euclid's Elements are all exhibited synthetically; the student is only employed in examining the soundness of the reasoning by which each successive addition is made to the collection of geometrical truths already obtained; and there is no hint given as to the manner in which the propositions were originally discovered. Some of the constructions and demonstrations appear rather artificial, and we are thus naturally induced to enquire whether any rules can be discovered by which we may be guided easily and naturally to the investigation of new propositions.

14. Geometrical analysis has sometimes been described in language which might lead to the expectation that directions could be given which would enable a student to proceed to the demonstration of any proposed theorem, or the solution of any proposed problem, with confidence of success; but no such directions can be given. We will state the exact extent of these directions. Suppose that a new theorem is proposed for investigation, or a new problem for trial. Assume the truth of the theorem or the solution of the problem, and deduce consequences from this assumption combined with results which have been already established. If a consequence can be deduced which contradicts some result already established, this amounts to a demonstration that our assumption is inadmissible; that is, the theorem is not true, or the problem cannot be solved. If a consequence can be deduced which coincides with some result already established, we cannot say that the assumption is inadmissible; and it *may happen* that by

starting from the consequence which we deduced, and retracing our steps, we can succeed in giving a synthetical demonstration of the theorem, or solution of the problem. These directions, however, are very vague, because no certain rule can be prescribed by which we are to combine our assumption with results already established; and moreover no test exists by which we can ascertain whether a valid consequence which we have drawn from an assumption will enable us to establish the assumption itself. That a proposition may be false and yet furnish consequences which are true, can be seen from a simple example. Suppose a theorem were proposed for investigation in the following words; *one angle of a triangle is to another as the side opposite to the first angle is to the side opposite to the other.* If this be assumed to be true we can immediately deduce Euclid's result in I. 19; but from Euclid's result in I. 19 we cannot retrace our steps and establish the proposed theorem, and in fact the proposed theorem is false.

Thus the only definite statement in the directions respecting geometrical analysis is, that if a consequence can be deduced from an assumed proposition which contradicts a result already established, that assumed proposition must be false.

15. We may mention, in particular, that a consequence would contradict results already established, if we could shew that it would lead to the solution of a problem already given up as impossible. There are three famous problems which are now admitted to be beyond the power of Geometry; namely:

To find a straight line equal in length to the circumference of a given circle,

To trisect any given angle, and

To find two mean proportionals between two given straight lines.

The grounds on which the geometrical solution of these problems is admitted to be impossible cannot be explained without a knowledge of the higher parts of mathematics; the student may, however, be content with the fact that innumerable attempts have been made to obtain solutions, and that these attempts have been made in vain.

The first of these problems is usually referred to as the *Quadrature of the Circle.* For the history of it the student should consult the article in the *English Cyclopædia* under that head, and also a series of papers in the *Athenæum* for 1863 and subsequent years, entitled a *Budget of Paradoxes*, by Professor De Morgan.

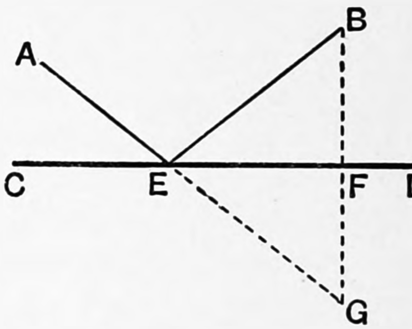
The third of the three problems is often referred to as the *Duplication of the Cube.*

We will now give some examples of geometrical analysis.

16. *From two given points it is required to draw to the same point in a given straight line two straight lines equally inclined to the given straight line.*

Let A and B be the given points, and CD the given straight line.

Suppose AE and EB to be the two straight lines equally inclined to CD. Draw BF perpendicular to CD, and produce AE and BF to meet at G.



Then the $\angle$ BED = the $\angle$ AEC, by hypothesis; and

the $\angle$ AEC = the $\angle$ DEG. [I. 15.

Hence the $\triangle^s$ BEF and GEF are equal in all respects; [I. 26.

$\therefore$ FG = FB.

This result shews how we may synthetically solve the problem. Draw BF perpendicular to CD, and produce it to G, so that FG may be equal to FB; then join AG, and AG will intersect CD at the required point.

If A be on the opposite side of CD from B, join GA, and let it be produced to meet CD in K. Then AK, BK are equally inclined to BD.

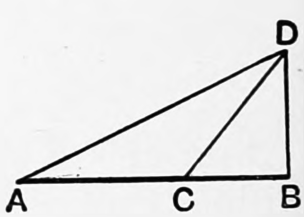
17. *To divide a given straight line into two parts such that the difference of the squares on the parts may be equal to a given square.*

Let AB be the given straight line, and suppose C the required point.

Then $AC^2 - CB^2$ = given sq.; $\therefore$ $AC^2 = CB^2$ + given square.

Erect at AB a perpendicular BD so that

BD^2 = given sq.



$\therefore$ $AC^2 = CB^2 + BD^2 = CD^2$; [I. 47.

$\therefore$ AC = CD, and the $\triangle$ ACD is isosceles.

The $\angle$ CAD therefore = the $\angle$ ADC.

Hence we have the following synthetical construction. At B draw BD at right $\angle^s$ to AB and equal to the side of the given sq. Join AD, and make the $\angle$ ADC equal to the $\angle$ BAD, and let DC meet AB in C. Then C is the required point, as may be easily shewn.

For $AC^2 = CD^2 = CB^2 + BD^2$.

It is obvious that the given square must not exceed the square on AB, in order that the problem may be possible.

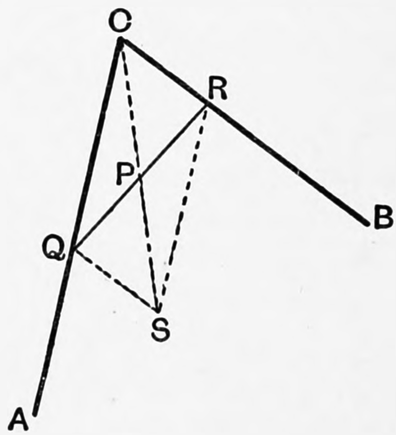
There are two positions of C, if it is not specified which of the two segments AC and CB is to be greater than the other; but only one position, if it is specified.

In like manner we may solve the problem, *to produce a given straight line so that the square on the whole straight line made up of the given straight line and the part produced, may exceed the square on the part produced by a given square, which is not less than the square on the given straight line.*

The two problems may be combined in one enunciation thus, *to divide a given straight line internally or externally so that the difference of the squares on the segments may be equal to a given square.*

18. *Given two intersecting straight lines,* OA *and* OB, *and any point* P *between them; to draw through* P *a straight line such that the portion of it intersected between* OA *and* OB *is bisected at* P.

Let QPR be the required line. Join OP and produce to S so that OP = PS. Then QR and OS bisect one another at P, and this is a property of the diagonals of a $\|^{m}$. Hence



Construction. Join OP and produce it to S so that OP = PS. Draw $\|^{ls}$ SQ, SR as in the figure. Then SQOR is a $\|^{m}$;

∴ SO, QR bisect one another.

But P is the middle point of OS.

∴ P is the middle point of QR.

∴ QPR is drawn as required.

19. *Bisect a triangle* ABC *by a straight line drawn through a given point* P *in the base* BC.

Let PE be the required straight line.
Then area ABPE = half the $\triangle$ ABC.
But, if D be the middle point of BC,

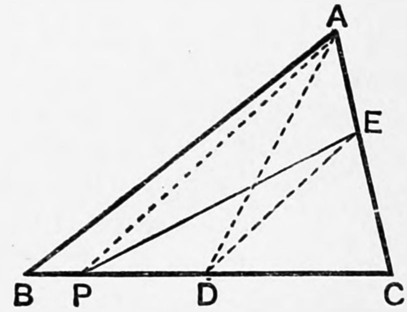
$\triangle$ ADB = half the $\triangle$ ABC. [I. 38.

∴ ABPE = $\triangle$ ADB.

∴ subtracting $\triangle$ ABP from each, then

$\triangle$ AEP = $\triangle$ ADP.

∴ AP, DE are parallel. [I. 39.

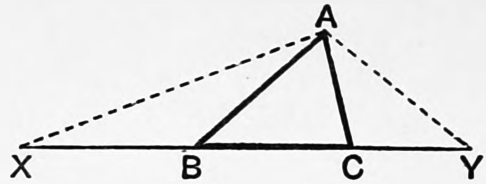


Hence the construction required: *Join* AP; *through* D, *the middle point of* BC, *draw* DE $\|^{l}$ *to* PA *to meet* AC *in* E; *then* PE *is the required straight line.*

For AEPB = $\triangle$ AEP + $\triangle$ ABP = $\triangle$ ADP + $\triangle$ ABP = $\triangle$ ADB

= half of $\triangle$ ABC.

20. *Construct a triangle, given its perimeter and its base angles.*



Suppose that ABC is the required △ ; produce CB to X and BC to Y so that BX = BA and CY = CA. Then XY is equal to the given perimeter, and ∠BXA = ∠BAX, and ∴ ∠ABC = twice ∠AXY.

∴ ∠AXY = half ∠ABC, and similarly ∠AYX = half ∠ACB.

Hence the construction required ; Let XY equal the given perimeter ; at X make ∠YXA = half one given base ∠, and at Y make ∠XYA = half the other given base ∠. At A make ∠XAB = ∠AXB and ∠YAC = ∠AYC where AB, AC lie between AX, AY and meet XY in B, C. Clearly ABC is now the required △.

EXERCISES.

1. Construct an isosceles triangle having given the base and the perpendicular on it from the opposite angle.

Construct a right-angled triangle having given

2. the hypotenuse and an acute angle ;

3. the hypotenuse and a side ;

4. the hypotenuse and the sum of the remaining sides ;

5. the hypotenuse and the difference of the other sides ;

6. the hypotenuse and the perp^r on it from the right angle. [Use Page 60, Ex. 8.]

7. a side and the perpendicular on the hypotenuse from the right angle;

8. the perimeter and an acute angle. [Use Art. 20.]

9. Construct a triangle having given the middle points of its sides.

10. Construct a triangle, given its three medians.

[Let ABC be a △, G its centroid and AD the median through A ; complete the ||^gm BGCO ; then GO is bisected at D and GCO is a △ having its sides respectively equal to two-thirds of the medians. Hence the construction.]

11. Construct a triangle, given the base, the difference of the angles at the base, and the difference of its sides.

[Let BC be the given base ; make ∠BCD = $\frac{1}{2}$ the given diff. of the base ∠^s, and take D on CD such that BD = the given difference of the sides ; produce BD to A, making ∠ACD = ∠ADC. Then ABC is the required △.]

12. Construct a triangle, given its base, one of its base angles, and (1) the sum, (2) the difference of its sides.

13. Bisect a parallelogram by a straight line drawn through a given point in its plane.

14. Trisect a right angle.

15. Bisect a quadrilateral by a straight line drawn through (1) an angular point, (2) a point in one of the sides.

[(1) Let ABCD be the quad[l], B being nearer to AC than D; through E, the middle point of BD draw a line ||[l] to AC to meet DC in G; then AG bisects the quad[l]. (2) Let the given point be P in AB; draw DE ||[l] to AC to meet BA in E; through F, the mid-point of BE, draw FQ ||[l] to CP to meet CD in Q; then PQ bisects the quad[l].]

16. Construct a parallelogram, given one side and the two diagonals.

17. Trisect a given straight line.

[On the given straight line AB describe an equilateral △ ABC, and through its centroid G draw GX, GY ||[l] to AC, CB to meet AB in X, Y.]

18. Trisect a triangle by a straight line drawn through a point in one side. [Use I. 44 to first cut off one third of the triangle by a straight line through the given point, and then use Ex. 15 (1).]

19. Trisect a triangle by a straight line through an angular point.

[Use Ex. 17.]

20. On the side AB of a parallelogram ABCD describe a triangle equal in area to ABCD and having the ∠ at A common.

21. Construct a triangle of given area with two sides of given length.

22. Inscribe a square of given magnitude in a given square.

23. Trisect a parallelogram by straight lines drawn through one angular point. [By Ex. 17 trisect the sides of the parallelogram opposite the given angular point.]

24. Describe a △ equal to a given quadrilateral. [See Page 68, Ex. 2.]

25. Describe a △ equal in area to a given rectilineal figure. [Use Ex. 24.]

ON LOCI.

21. A *locus* consists of all the points which satisfy certain conditions and of those points alone. Thus, for example, the locus of the points in a fixed plane which are at a given distance from a given point is the circumference of the circle described from the given point as centre, with the given distance as radius; for all the points on the circle, and no others, are at the given distance from the given point.

Again, the locus of all points which are at a given distance from a given straight line is one or other of the two straight lines parallel to the given straight line and at the given distance from the given straight line, one on one side and the other on the other side of it.

We shall restrict ourselves to loci which are situated in a fixed plane, and which are properly called *plane loci.*

22. Several of the propositions in Euclid furnish good examples of loci. Thus the locus of the vertices of all triangles which are on the same base and on the same side of it, and which have the same area, is a straight line parallel to the base; this is shewn in I. 37 and I. 39.

Again, the locus of the vertices of all triangles which are on the same base and on the same side of it, and which have the same vertical angle, is a segment of a circle described on the base; for it is shewn in Page 145, that all the points thus determined satisfy the assigned conditions, and it is easily shewn that no other points do.

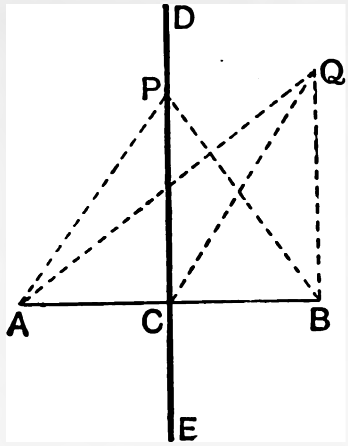
We will now give some examples. In each example we ought to shew not only that all the points which we indicate as the locus do fulfil the assigned conditions, but that no other points do. This second part however we shall generally leave to the student.

23. *Required the locus of points which are equidistant from two given points.*

Let A and B be the two given points. Join AB, and bisect it at C. Draw DCE at right angles to AB. Then DCE shall be the required locus. For take *any* point P on it and join PA and PB. Then in the $\triangle^s$ PCA, PCB we have PC common, AC=CB, and the right $\angle$PCA=the right $\angle$PCB;

$\therefore$ the base PA=the base PB, and $\therefore$ P is equidistant from A and B.

Also, no point out of DCE is equidistant from A and B. For, if

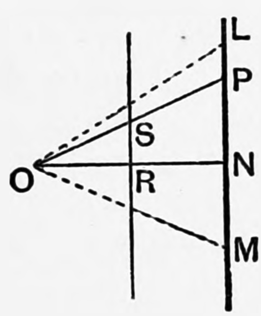


possible, let Q be such a point. Join QA, QC, QB. The sides of the triangles QCA, QCB are then all equal, each to each ;

∴ ∠QCA = ∠QCB ; ∴ each is a rt. ∠ ; ∴ Q lies on the st. line DCE.

This line is thus the required locus.

24. *Find the locus of the middle points of all straight lines drawn from a fixed point* O *to all points on a given straight line.*



Let LM be the given straight line. Draw ON perpendicular to it, and bisect ON at R. Take *any* point P on LM ; join OP, and bisect it in S. Join RS. Since R, S are the middle points of the sides ON, OP ; ∴ RS is parallel to NP ; [Art. 1.

∴ ∠ORS = ∠ONP = a right ∠ ; [I. 29.

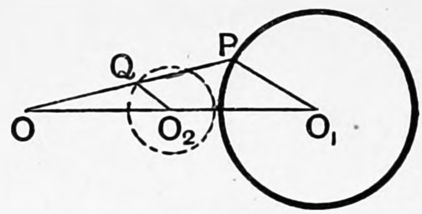
∴ S lies on the straight line bisecting ON at right angles.

Similarly, the middle point of the straight line joining O to any other point of LM lies on RS ;

∴ the required locus is the straight line bisecting ON at right angles ; it is therefore ||[l] to LM.

25. *The locus of the middle point of the straight lines drawn from a fixed point* O *to all points on a given circle is another circle.*

Let the given circle have O_1 as centre. Join O to any point P on the circle. Bisect OP in Q and OO_1 in O_2. Join O_2Q.



Since O_2, Q are the middle points of OO_1 and OP;

$\therefore$ O_2Q is ||^l to and one half of O_1P; [Art. 1.

$\therefore$ O_2Q is constant and O_2 is a fixed point;

$\therefore$ Q always lies on a fixed circle whose centre is O_2, and whose radius is one-half that of the given circle.

26. *Required the locus of the vertices of all triangles on a given base* AB, *such that the square on the side terminated at* A *may exceed the square on the side terminated at* B, *by a given square.*

Suppose C to denote a point on the required locus; from C draw CD perpendicular to AB, produced if necessary.

Then sq. on AC = sqs. on AD, DC,

and sq. on BC = sqs. on BD, DC; [I. 4.

$\therefore$ sq. on AC − sq. on BC = sq. on AD − sq. on BD;

that is, given sq. = sq. on AD − sq. on BD;

$\therefore$ D is a fixed point either in AB or in AB produced through B. [Art. 17.

Also CDB is a right $\angle$.

The required locus is the straight line drawn through the known point D, at right angles to AB.

EXERCISES ON LOCI (BOOK I.).

1. The locus of points equally distant from two given intersecting straight lines is one or other of two fixed straight lines.

2. What is the locus of points equidistant from two parallel straight lines?

3. Find the locus of the middle points of all straight lines parallel to the base of a triangle and terminated by the sides.

4. From any point on the base of a triangle straight lines are drawn parallel to the sides; the locus of the point of intersection of the diagonals of the ||^{gms} so formed is a straight line parallel to the base.

5. The locus of all points, the sum of whose perpendicular distances

from two given perpendicular straight lines is given, is a straight line which is equally inclined to the two given lines.

6. Find the locus of all points the difference of whose perpendicular distances from two given straight lines is given.

7. Find the locus of a point P such that the sum of the areas of the triangles PAB, PCD with given bases AB, CD is constant.

[Let AB, CD meet in O; take E, F on OA, OC so that OE = AB and OF = CD. Then $\triangle$ PAB = $\triangle$ POE and $\triangle$ PCD = $\triangle$ POF;

$\therefore$ area OEPF is const. But area OEF is the same for all positions of P:

$\therefore$ $\triangle$ EPF is of constant area;

$\therefore$ locus of P is a straight line parallel to EF.]

8. AB and CD are two straight lines given in magnitude and position, and a point P moves so that the $\triangle$s PAB, PCD are equal in area; the locus of P is a straight line passing through the intersection of AB and CD, produced if necessary.

[As in the previous solution we have $\triangle$ OPE = $\triangle$ OFP. Hence if OP meet EF in N and EL, FM be perprs to OP we have EL = FM, and $\therefore$ EN = NF; $\therefore$ P lies on the straight line joining O and the middle point of EF.]

9. Find the locus of the points of intersection of the diagonals of a parallelogram whose base and area are given.

10. Find the locus of the intersection of the medians of all triangles whose base and area are given.

11. The locus of the vertices of all right-angled triangles described upon a given straight line as hypotenuse is a circle.

12. A ladder is raised gradually, its ends being always in contact with a vertical wall and the ground respectively; the locus of its centre is a circle.

13. The locus of the vertex A of a triangle ABC, given the base BC and the length of the median through B, is a circle.

14. Find a point in a given straight line which is equidistant from (1) two given points, and (2) two given straight lines.

15. Find a point which is equidistant from three points which are not in the same straight line.

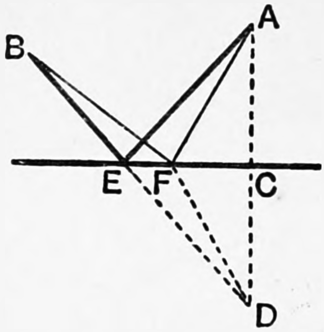
16. If two opposite ends of a square of given magnitude move along two straight lines which are at right angles, then the remaining angles move along two perpendicular straight lines.

17. Construct a triangle having given the base, the altitude, and the length of the median which bisects the base.

18. Construct an isosceles triangle which has the same base and the same area as a given triangle.

MAXIMA AND MINIMA.

27. *In a given indefinite straight line it is required to find a point such that the sum of its distances from two given points on the same side of the straight line shall be the least possible.*



Let A and B be the two given points. From A draw AC perpendicular to the given straight line and produce AC to D so that CD = AC. Join DB meeting the given straight line at E.

Then E *shall be the required point.*

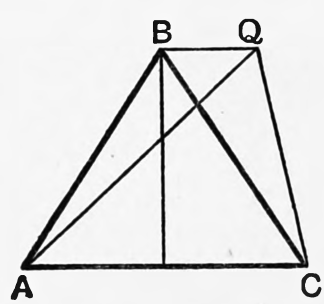
For let F be any other point in the given straight line. Then, because AC = DC, and EC is common to the two △s ACE, DCE, and that the right ∠ACE = the right ∠DCE, ∴ AE = DE. [I. 4.

Similarly, AF = DF. Also the sum of DF and FB > BD. [I. 20.

∴ the sum of AF and FB > BD;

that is, the sum of AF and FB > the sum of AE and EB.

28. *The perimeter of an isosceles triangle is less than that of any other triangle of equal area standing on the same base.*



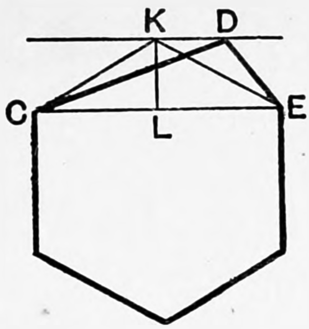
Let ABC be an isosceles triangle; QBC any other triangle equal in area and standing on the same base BC.

Join AQ; then BQ is parallel to AC. [I. 39.

It follows from Art. 27 that the sum of AQ and QC is greater than the sum of AB and BC.

Cor. Of all △s of the same perimeter that which has the greatest area is isosceles.

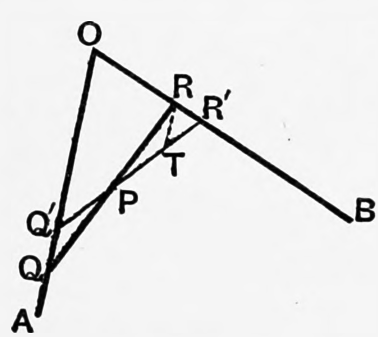
29. *If a polygon be not equilateral a polygon may be found of the same number of sides, and equal in area, but having a less perimeter.*



For, let CD, DE be two adjacent unequal sides of the polygon. Join CE. Through D draw a straight line parallel to CE. Bisect CE at L; from L draw a straight line at right angles to CE meeting the straight line drawn through D at K. Then by removing from the given polygon the triangle CDE and applying the triangle CKE, we obtain a polygon having the same number of sides as the given polygon, and equal to it in area, but having a less perimeter. [Art. 28.

30. *Through a given point* P *draw a straight line which shall cut off from two given straight lines* OA, OB *a triangle of minimum area, and shew that the required straight line is bisected at* P.

Draw through P a straight line QR which is bisected at P. [Art. 18.



Then QOR shall be the required minimum $\triangle$.

For draw any other line Q′PR′ to cut off the $\triangle$ OQ′R′.

Draw RT parallel to OQ′ to meet Q′PR′ in T.

Then $PR=PQ$, $\angle TPR=\angle Q'PQ$, and $\angle PRT=\angle PQQ'$. [I. 29.

$$\therefore \triangle PQQ' = \triangle PTR.$$

$$\therefore \triangle PRR' > \triangle PQQ'.$$

Add to each the area ROQ′P. $\therefore \triangle Q'OR' > \triangle QOR$.

$\therefore$ any other straight line through P cuts off a greater $\triangle$ than QR cuts off.

$\therefore$ QOR is the required minimum $\triangle$.

In this question it will be noticed that the position of QR, when it cuts off the minimum area, is a **symmetrical** one, in that it is bisected at P. We shall find that this characteristic of being a symmetrical position is a general property of lines, areas, and angles, which are either maxima or minima.

Thus in Art. 27, if F is to the right of E, the $\angle AFC > \angle BFE$; if it be to the left, then $\angle AFC <$ the corresponding angle; it is only when F is at the critical, or turning, point E that these angles are just equal.

Similarly in Art. 28, if Q be anywhere on AQ to the right or the left of A, the sides AQ, QC are not equal, they are only equal when Q is at the critical point B.

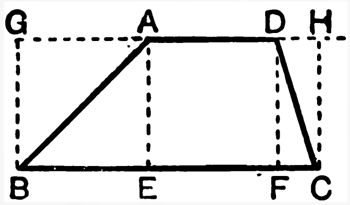
EXERCISES.

1. Find the triangle of greatest area, two of its sides being given.

2. Find the parallelogram of greatest area, two of its adjacent sides being given.

THEOREMS AND EXAMPLES ON BOOK II.

31. *The area of a trapezium equals half the rectangle contained by the sides and the perpendicular distance between them.*



Let ABCD be the trapezium with AD, BC the parallel sides. Draw perp^rs AE, DF, BG, CH on the opposite sides.

Then $\triangle$ ABE = half the rect. EG,

and $\triangle$ DFC = half the rect. FH. [I. 41.

$\therefore$ twice $\triangle$ ABE + twice $\triangle$ DFC + twice rect. AF

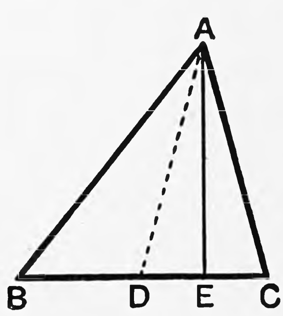
= rect. EG + rect. FH + twice rect. AF

= rect. BH + rect. AF = rect. AE, BC + rect. AD, AE.

that is, area ABCD

= half the rect. contained by AE and the sum of BC, AD.

32 *If* ABC *be a triangle,* D *the middle point of* BC *and* E *the foot of the perpendicular from* A *upon* BC, *the difference of the squares on* AB, AC = *twice the rect.* BC, DE.



We have $AB^2 = AE^2 + BE^2$,

and $AC^2 = AE^2 + CE^2$,

$\therefore$ the diff. of the sqs. on AB, AC

= the diff. of the sqs. on BE, EC

= the rect. contained by the sum and difference of BE, EC. [II. 5, Note.

Now the sum of BE, EC = BC, and

their difference = BE − EC = BE + DE − DC = BD + 2DE − DC

= 2DE, since D is the middle pt. of BC.

$\therefore$ diff. of sqs. on AB, AC = 2 rect. BC, DE.

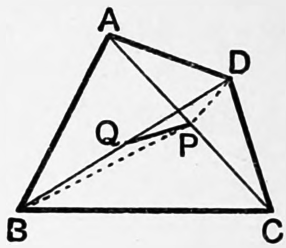
This proposition may be enunciated thus: *The difference of the squares on the sides of a* $\triangle$ = *twice the rect. contained by the base and the projection on the base of the median through the vertex.*

It has already been shewn [II. 13, Exercise I.] that

$$AB^2 + AC^2 = 2AD^2 + 2DB^2.$$

33. ABCD *is any quadrilateral and* P, Q *the middle points of its diagonals* AC, BD ; *then the sum of the squares on its sides exceeds the squares on its diagonals by four times the square on the line joining the middle points of the diagonals, i.e.*

$$AB^2+BC^2+CD^2+DA^2=AC^2+BD^2+4PQ^2.$$



For [by the previous theorem]

$$AB^2+BC^2=2AP^2+2BP^2,$$

and $AD^2+DC^2=2AP^2+2DP^2$.

$$\therefore\ AB^2+BC^2+CD^2+DA^2=4AP^2+2BP^2+2DP^2$$
$$=AC^2+2BP^2+2PD^2.$$

Also $BP^2+PD^2=2PQ^2+2DQ^2$

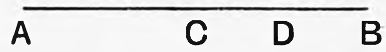
$$\therefore\ 2BP^2+2PD^2=4PQ^2+4DQ^2=4PQ^2+BD^2.$$
$$\therefore\ AB^2+BC^2+CD^2+DA^2=AC^2+BD^2+4PQ^2.$$

In the particular case when ABCD is a ||gm P and Q coincide, and ∴ PQ vanishes. The proposition then is; *The sum of the sqs. on the sides of a parallelogram = sum of the sqs. on the diagonals.*

34. *If a straight line* AB *is divided into two parts at* D,

(1) *the sum of the squares on* AD, DB *is least,*

and (2) *the rect.* AD, DB *is greatest, when the straight line is bisected.*



Let C be the middle point of AB.

(1) By II. 9 we have $AD^2+DB^2=2AC^2+2CD^2$.

Now AC is of the same length wherever the point D is.

∴ $2AC^2+2CD^2$ is least when CD is least, and this is when CD is zero, *i.e.* when D coincides with C.

∴ AD^2+DB^2 is least when D is at C.

(2) By II. 5 we have rect. AD, $DB+CD^2=AC^2=$ a constant for all positions of D.

∴ rect. AD, DB is greatest when CD^2 is least, *i.e.* when CD is zero, *i.e.* when D coincides with C.

Corollary. Since, by (2), the rectangle contained by two straight lines, AD, DB, whose sum is given, is greatest when they are equal, it follows that *Of all rectangles with a given perimeter the greatest is a square.*

EXERCISES.

****1.** The locus of the vertex of a triangle on a given base, when the sum of the squares on its sides is given, is a circle.

[For, in the figure of Art, 32, DA is constant and known.]

****2.** When the difference of the squares is given, the locus is a straight line.

[For in this case DE is constant and known.]

****3.** The base BC of a triangle ABC is divided at D, so that

$$m \cdot BD = n \cdot CD;$$

prove that

$$m \cdot AB^2 + n \cdot AC^2 = m \cdot BD^2 + n \cdot DC^2 + (m+n) AD^2.$$

If D be in BC produced, then

$$m \cdot AB^2 - n \cdot AC^2 = m \cdot BD^2 - n \cdot DC^2 + (m-n) \cdot AD^2.$$

[Use II. 12 and 13.]

****4.** Given the base of a △ in magnitude and position, and the sum, or difference, of m times the square on one side and n times that on the other, prove that the locus of the vertex is a circle.

[Use the previous Exercise.]

****5.** If the medians of a triangle ABC meet in G, prove that

$$AB^2 + BC^2 + CA^2 = 3(GA^2 + GB^2 + GC^2).$$

****6.** If G be the centroid of a △ ABC and P any point in its plane, prove that $PA^2 + PB^2 + PC^2 = GA^2 + GB^2 + GC^2 + 3 \cdot PG^2$. [Use Ex. 3.]

7. The locus of a point, which moves so that the sum of the squares of its distances from the angular points of a triangle is constant, is a circle whose centre is the centroid of the triangle. [Use Ex. 6.]

8. Find a point in the plane of a triangle such that the sum of the squares of its distances from the angular points is a minimum.

[Use Ex. 6.]

9. Find a point in a given straight line such that the sum of the squares of its distances from two given points is a minimum.

Divide a given straight line AB at C so that

10. $AB \cdot BC$ = given square. [Use I. 44.]

11. $AC^2 = 2BC^2$.

12. $AB^2 + BC^2 = 2AC^2$.

Produce a given straight line AB to C, so that

13. $AB^2 + AC^2 = 2AC \cdot BC$.

14. $AB^2 + BC^2 = 2AC \cdot BC$.

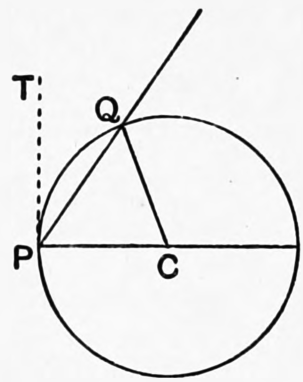
15. $AC \cdot BC = AB^2$. [BC = AH, Fig. II., 11.]

16. $AC \cdot AB$ = a given square.

THEOREMS AND EXAMPLES ON BOOK III.

DEFINITION OF A TANGENT AS A LIMIT.

35. Another definition of the tangent is often much more useful than Euclid's.



If PQ be a straight line through a point P on a circle, of centre C, which meets it again in Q, then PQ is called **a secant.**

If P be kept fixed and Q move along the circle until it coincides with P the limiting position of PQ, when Q becomes indefinitely close to P, is called the **tangent** at P.

[If the student conceive the circumference of the circle as made up of infinitely small dots, packed infinitely close together, then this method of looking upon the tangent at P conceives it as the line joining P and the very next dot to P.]

36. We can easily shew that the tangent PT is perpendicular to PC. For since $CQ = CP$, the $\angle CPQ = \angle CQP$.

$\therefore$ twice $\angle CPQ = \angle CPQ + \angle CQP = 2$ rt. $\angle^s - \angle PCQ$. [I. 32.

This holds for all positions of Q. When Q is indefinitely close to P, the $\angle PCQ$ vanishes, and then in the limiting position twice the $\angle CPQ$ is two right $\angle^s$.

But in the limit the $\angle CPQ$ becomes the $\angle CPT$,

$\therefore$ twice the $\angle CPT =$ two right $\angle^s$,

i.e. $\angle CPT =$ a right $\angle$.

We can also deduce III. 32. For let Q be a point close to B, in the figure of that proposition, and produce BQ to R. Then BQDA is a cyclic quadrilateral, and

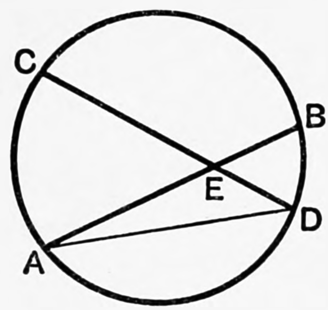
$\therefore \angle BAD =$ two rt. $\angle^s - BQD$ [III. 22.

$= \angle DQR$.

Now let Q become indefinitely close to B, then BQR ultimately coincides with BF, and

$\therefore$ $\angle$BAD = the ultimate position of $\angle$DQR = $\angle$DBF

37. *If two chords intersect within a circle, the angle which they include is measured by half the sum of the intercepted arcs.*



Let the chords AB and CD intersect at E; join AD.

The $\angle$AEC = the $\angle$s ADE, DAE, [I. 32.

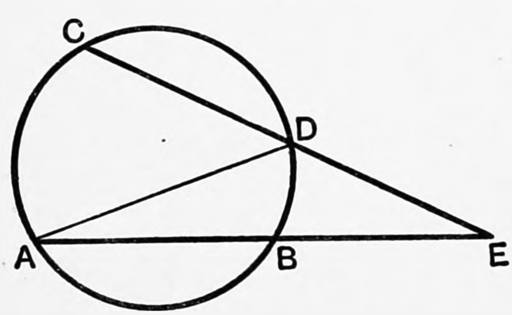
that is, = the $\angle$s standing on the arcs AC, BD,

that is, = the $\angle$ at the *circumference* standing on the sum of the arcs AC, BD,

that is, = the $\angle$ at the *centre* standing on half the sum of the arcs AC, BD. [III. 20.

Similarly the angle CEB is measured by half the sum of the arcs CB and AD.

38. *If two chords produced intersect without a circle, the angle which they include is measured by half the difference of the intercepted arcs.*



Let the chords AB and CD, produced, meet in E; join AD.

The $\angle$ADC = the $\angle$s EAD, AED, [I. 32.

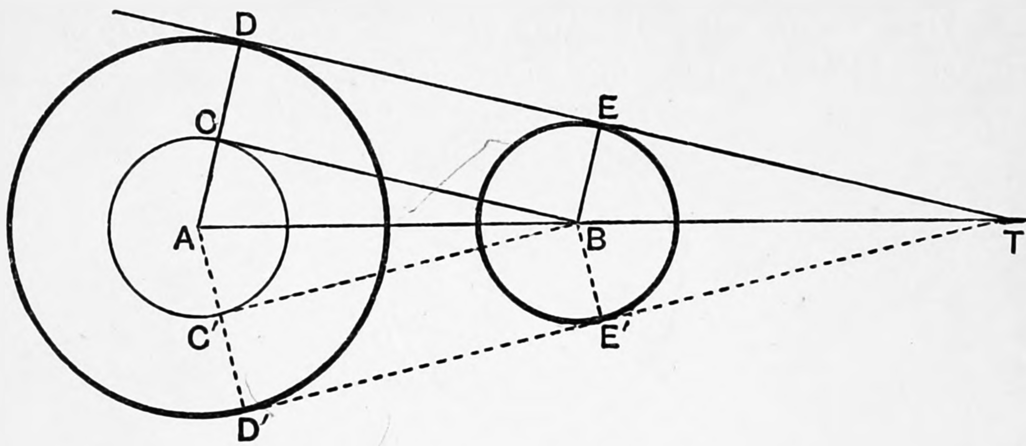
that is, $\angle$AEC = difference of the $\angle$s ADC, BAD,

that is, = $\angle$ at *circumference* standing on the difference of the arcs AC, BD,

and $\therefore$ = $\angle$ at *centre* standing on half the difference of these arcs.

39. *To draw a common tangent to two given circles.*

Let A be the centre of the greater circle, and B the centre of the less circle. With centre A, and radius equal to the difference of the



radii of the given circles, describe a circle; from B draw a tangent BC to the circle so described.

Join AC and produce it to meet the circumference at D.

Draw the radius BE parallel to AD, and on the same side of AB; and join DE.

Then DE *shall touch both circles.*

For BE and CD are equal and parallel lines. [*Construction.*

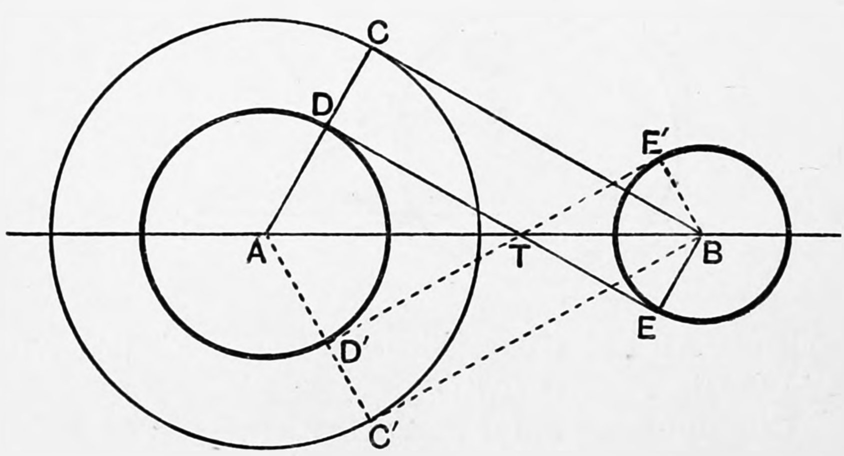
∴ BC and DE are equal and parallel. [I. 33.

∴ ∠ADE = ∠ACB [I. 29] = a right ∠.

∴ since BEDC is a $\|^m$,

∠BED = ∠BCD = a right ∠.

∴ DE touches both circles at D, E. [III. 16, *Corollary.*



Since two tangents can be drawn from B to the described circle, two solutions can be obtained; and the two straight lines which are thus

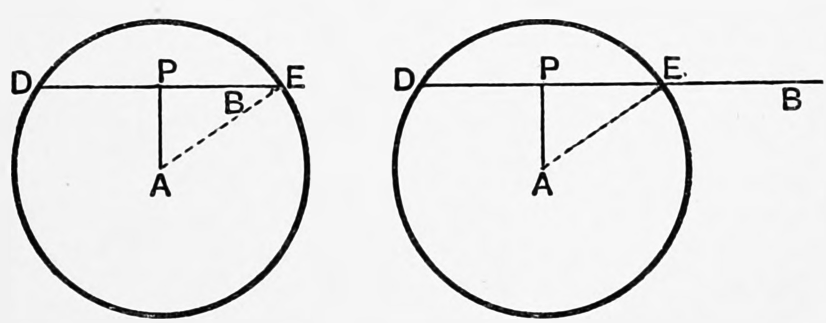
drawn to touch the two given circles can be shewn to meet AB, produced through B, at the same point T. The construction is applicable when each of the given circles is without the other, and also when they intersect.

When each of the given circles is without the other, we can obtain two other solutions. For, as in the second figure, describe a circle with A as centre and radius equal to the sum of the radii of the given circles; and continue as before, except that BE and AD will now be on *opposite* sides of AB. The two straight lines which are thus drawn to touch the two given circles can be shown to intersect AB at the same point. [See Art. 84.]

LOCI [BOOK III.].

40. *Required the locus of the middle points of all the chords of a circle which pass through a fixed point.*

Let A be the centre of the given circle; B the fixed point; let any



chord DE of the circle be drawn so that, produced if necessary, it may pass through B. Let P be the middle point of this chord, so that P is a point on the required locus.

Since P is the middle point of the chord DE,

$\therefore$ $\angle$APE, that is $\angle$APB, is a right $\angle$. [III. 3.

$\therefore$ P is on a circle of which AB is a diameter. Hence if B be within the given circle the locus is the circle described on AB as diameter; if B be without the given circle the locus is that part of the circle described on AB as diameter, which is within the given circle.

41. *A chord of given length slides round with its ends on the circumference of a given circle; the loci of*
(1) *its middle point,*
(2) *any fixed point on it, are circles.*

Let DE [Fig. Art. 40] be the chord of given length, P its middle point; A the centre.

Then $\angle$APE is a rt. $\angle$. [III. 3.

$\therefore$ $AP^2 = AE^2 - PE^2 =$ a constant, since the radius of the circle and the length of the chord are given.

$\therefore$ AP is constant, and hence P moves on a circle whose centre is A.

(2) If B be the fixed point on the chord, then PB is constant.

$\therefore$ $AB^2 = AP^2 + PB^2 =$ constant.

$\therefore$ AB is constant, and the locus of B is a circle whose centre is A.

EXERCISES.

1. Find the locus of the middle points of parallel chords of a circle.

2. Find the locus of the middle points of equal chords of a circle.

3. Find the locus of the point of intersection of tangents to a given circle which meet at a given angle.

4. Find the locus of all points from which tangents to a given circle are equal.

5. Find the locus of the points from which the tangents drawn to a given circle are of given length.

6. Find the locus of the point of contact of all tangents drawn from a given point to a system of concentric circles.

7. Find the locus of the vertex of a triangle whose base BC is given and of which the median through B is of given length.

8. The locus of the middle point of a straight line of constant length, whose ends move on two fixed perpendicular straight lines, OA and OB, is a circle whose centre is O.

9. Given the base and vertical angle of a triangle; find the locus of the middle point of the straight lines joining the vertices of all such triangles to the middle point of the base.

10. Given the base BC of a triangle and its vertical angle, prove that the loci of

(1) the intersection of the bisectors of the base $\angle^s$,
(2) the intersection of the bisectors of the exterior base $\angle^s$,
(3) the intersection of the bisector of one exterior base $\angle$ with the bisector of the other interior base $\angle$,

are all arcs of circles passing through B and C.

11. If two segments of circles have a common chord AB and any two points P and Q be taken, one on each segment, prove that the locus of O, the point of intersection of the bisectors of the angles PAQ, PBQ, is another segment of a circle passing through A and B.

[Twice $\angle AOB = 4$ rt. $\angle^s - 2\angle OAB - 2\angle OBA$

$$= 4 \text{ rt. } \angle^s - \angle PAB - \angle QAB - \angle PBA - \angle QBA$$

$$= \angle APB + \angle AQB = \text{const.} \quad \therefore \text{ etc.}]$$

12. If A and B be two fixed points on a circle and C, D the extremities of a chord of constant length, then the intersection of AD, BC and also that of AC, BD lie on fixed circles.

13. O is a fixed point on the circumference of a circle and OP is any chord. On OP is described a circle containing a given angle; the locus of its centre is one or other of two fixed circles.

14. PQ is a straight line of given length which moves so that its ends, P and Q, slide on two given straight lines OA and OB; the locus of the intersection of perpendiculars to OA and OB, drawn through P and Q respectively, is a circle.

15. If a chord of a circle, centre O, subtend a right angle at a fixed point P, the locus of its middle point is a circle whose centre is the middle point of PO.

[If QR be the chord, and C, T be the middle points of OP, QR then by Page 109, Ex. 1, $4OC^2 + 4CT^2 = 2OT^2 + 2TP^2 = 2OQ^2 - 2TR^2 + QP^2 + PR^2 - 2TR^2 = 2OQ^2 - 4TR^2 + QR^2 = 2OQ^2$, etc.]

Find the locus of the centres of all circles which

16. pass through two given points.

17. touch two given intersecting straight lines.

18. touch a given straight line at a given point.

19. touch a given straight line and are of given radius.

20. touch a given circle at a given point.

21. touch a given circle and are of given radius.

22. A triangle is formed by a fixed tangent to a circle, a variable tangent, and the chord joining their points of contact. Find the locus of the centre of its circumscribing circle.

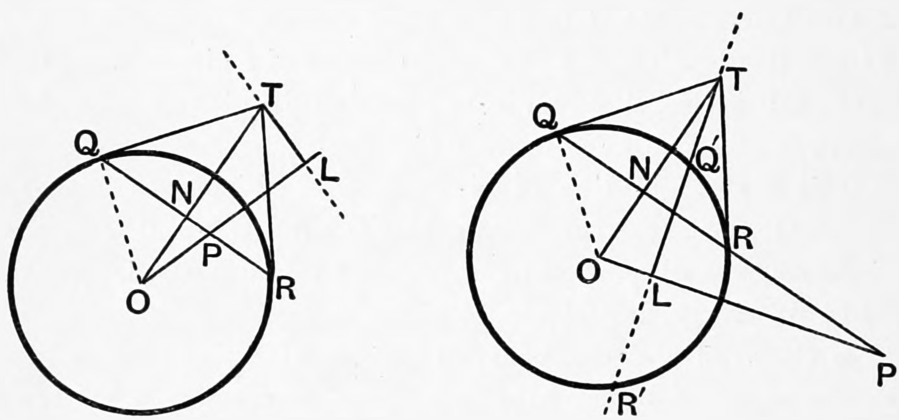
23. Tangents are drawn, one to each of two concentric circles, and the angle between them is constant. Prove that the locus of their point of intersection is one or other of two fixed circles concentric with the given one.

24. Two adjacent corners A, B of a sheet of paper are doubled down and meet at a point P in such a manner that the three parts of the edge AB form a triangle right-angled at P. Prove that the locus of P is an arc of a circle.

POLE AND POLAR.

42. The Polar of any point P with respect to a circle is the locus of the point of intersection of tangents drawn at the extremities of any chord of the circle which passes through P.

To find the polar of any given point P *and to construct it geometrically.*



Through P draw any straight line to cut the circle in Q and R. At Q and R draw tangents and let them meet in T. [III. 17.
Join OP and draw from T a perpendicular TL on OP or OP produced. Join OT and let it meet QR in N.
Then OQT is a $\triangle$ having OQT a right $\angle$, and QN is perpendicular to the base OT;

$$\therefore\ ON\ .\ OT = OQ^2 = (\text{radius})^2.$$

[For, by III. 13, $TQ^2 = TO^2 + OQ^2 - 2\,ON\ .\ OT$;

$$\therefore\ 2\,ON\ .\ OT = TO^2 + OQ^2 - TQ^2 = 2\,OQ^2 \ (\text{I. } 47).]$$

But since TNP, TLP are both right angles, a circle will go round T, N, P, L, and

$$\therefore\ OP\ .\ OL = ON\ .\ OT; \qquad [\text{III. } 36.$$

$$\therefore\ OP\ .\ OL = (\text{radius})^2;$$

$\therefore$ OL is constant, since P is a given point, and therefore OP constant;
$\therefore$ L is a fixed point and $\angle$OLT is a right angle;
$\therefore$ the locus of T (that is, the polar of P) is a straight line perpendicular to OL and passing through L.

The polar of P is thus a straight line, and may be constructed as follows:

On OP *take a point* L, *such that* OL . OP = (*radius*)^2, *and through* L *draw a straight line perpendicular to* OP.

The two points, L and P, are often called **inverse points**.

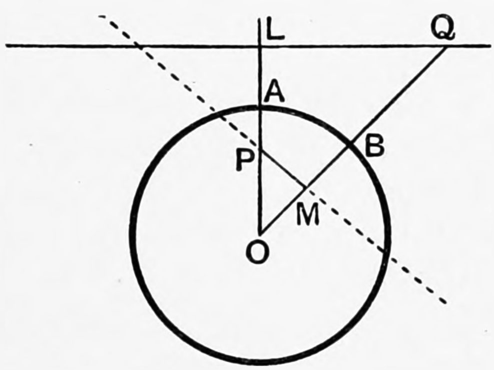
Corollary 1. If P be on the circle so also is L, and the polar of P in this case is the tangent at P.

Corollary 2. If, as in the right-hand figure, the line TL meet the circle in Q′ and R′ then the straight lines PQ′, PR′ would be tangents at Q′, R′.

For $OL \, . \, OP = (\text{radius})^2$, and Q′LR′ is perpendicular to OP.

Hence, if the point P lie outside the circle, its polar coincides with the line joining the points of contact of tangents drawn from it to the circle.

43. *If the polar of a point* P *passes through* Q *then the polar of* Q *passes through* P.



Let Q be any point on the polar LQ of a point P. Draw OPL perpendicular to LQ, O being the centre.

$$\text{Then } OP \, . \, OL = (\text{radius})^2. \qquad \text{[Art. 42.}$$

Join OQ and draw PM perpendicular to OQ.

Then since PLQ, PMQ are right angles, a circle will pass through P, L, Q, M;

$$\therefore \; OM \, . \, OQ = OP \, . \, OL \qquad \text{[III. 36.}$$
$$= (\text{radius})^2.$$

Hence, since OMP is a right angle, it follows, by Art. 42, that PM is the polar of Q,

that is, *the polar of* Q *passes through* P.

The polars of P *and* Q *meet at an angle equal to that subtended by* P *and* Q *at the centre of the circle.*

For, if O be the centre, the polar of P is perpendicular to OP,

and the polar of Q is perpendicular to OQ.

Also, the $\angle$ between two straight lines is equal to the angle between their perpendiculars;

$$\therefore \text{ the } \angle \text{ between the polars of P and Q} = \text{the } \angle POQ.$$

EXERCISES.

**1. The straight line joining two points, A and B, is the polar of the point of intersection of the polars of A and B.

[Let the latter intersect at T; then A lies on the polar of T, since T lies on the polar of A (43); so B lies on the polar of T; ∴ AB is the polar of T.]

**2. The point of intersection of any two straight lines is the pole of the straight line joining their poles.

**3. Find the locus of the poles of all straight lines which pass through a given point. [Use Art. 43.]

**4. A and B are two points in a plane of a circle whose centre is C; AX and BY are the perpendiculars from A and B on the polars of B and A respectively; prove that the rectangles CA . BY and CB . AX are equal. [Salmon's Theorem.]

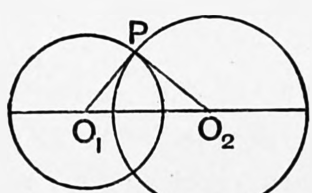
[Let the polar of A meet CA in M, and that of B meet CB in N; also draw AU, BV perpʳ to CB, CA respectively.

Then A, V, U, B lie on a circle; ∴ CU . CB = CA . CV. [III. 36.

But CN . CB = CM . CA = sq. on radius (Art. 42). Subtract;

∴ CB . NU = CA . MV, *i.e.* CB . AX = CA . BY.]

44. Orthogonal Circles. Def. Two circles are said to intersect orthogonally when the tangents at their points of intersection are at right angles.



If the two circles intersect at P, the radii O_1P and O_2P, which are perpendicular to the tangents at P, must also be at right angles.

Hence $$O_1O_2{}^2 = O_1P^2 + O_2P^2,$$

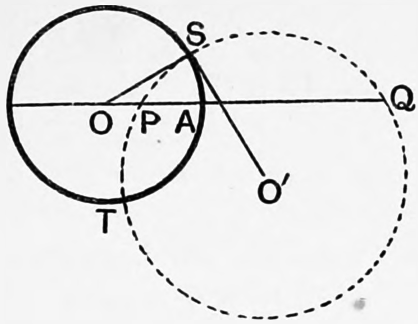
i.e. the square of the distance between the centres must be equal to the sum of the squares of the radii.

Also the tangent from O_2 to the other circle is equal to the radius a_2, *i.e. if two circles be orthogonal the length of the tangent drawn from the centre of one circle to the second circle is equal to the radius of the first.*

Either of these two conditions will determine whether the circles are orthogonal.

It follows that if we want the circle whose centre is O_2 which shall cut a given circle, centre O_1, orthogonally, we must take its radius equal to the tangent from O_2 to the given circle.

45. *Given a circle and two inverse points,* P *and* Q, *with respect to it; any circle which passes through* P *and* Q *cuts the given circle orthogonally.*



Let any such circle cut the given circle in S and T. Let O, O′ be the centres of the two circles and join OS, O′S. Then, by hypothesis, $OP \cdot OQ = OA^2 = OS^2$.

∴ OS touches the second circle at S. [III. 37.

∴ OSO′ is a right angle.

∴ the two circles cut orthogonally at S and similarly at T.

EXERCISES.

****1**. What is the locus of the centres of all circles which cut a given circle orthogonally at a given point?

****2**. Through a given point draw a circle to cut a given circle orthogonally at a given point.

****3**. If two circles cut orthogonally the extremities, P and Q, of any diameter of either, are conjugate points with respect to the other, *i.e.* the polar of P passes through Q.

[Let the centres of the circles be O and O′, and let them cut in C; let PO′Q be any diameter of the second; join OP and draw QS perp^r to OP; then S lies on the second circle. Since the circles cut othogonally the tangent at C to the circle O′ passes through O.

∴ $OS \cdot OP = OC^2 =$ sq. on radius of first circle;

∴ since PSQ is a right ∠, SQ is the polar of P with respect to the circle O (Art. 42); ∴ etc.]

4. P is any point in the plane of a circle C and Q any point on its polar with respect to C; the circle on PQ as diameter cuts C orthogonally.

5. All circles which pass through a given point and cut a given circle orthogonally pass through another given point.

[This is the converse of Art. 45.]

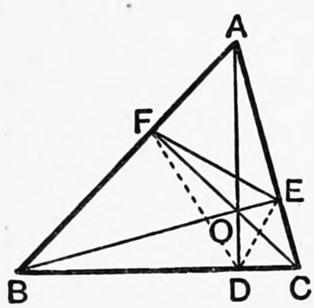
6. The chords joining two fixed points on a circle to the ends of any diameter intersect on a fixed circle which cuts the former orthogonally.

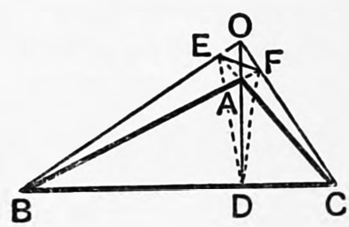
BOOK IV.

46. *The perpendiculars from the angles of a triangle on the opposite sides meet in a point.*

[A proof of this theorem has already been given in Art. 8].

Let ABC be a △ ; from B draw BE perpendicular to CA and CF perpendicular to AB; let BE, CF meet in O; join AO and produce it to meet BC in D; *then* AD *shall be perpendicular to* BC.





[If the △ be obtuse-angled, as in the second figure, some of these perpendiculars will meet the opposite sides produced].

Since AEO, AFO are right ∠s a circle will pass through A, E, O, F. [III. 22, *Converse.*

∴ ∠FAO = ∠FEO. [III. 21.

Similarly, a circle will pass through B, C, E, F.

∴ ∠FEO = ∠FCB;

∴ ∠FAO = ∠FCB, that is, ∠BAD = ∠FCB.

Also the ∠ at B is common to the △s BAD, BCF.

∴ the third ∠BDA = the third ∠BFC [I. 32.

= a right ∠. [*Construction.*

∴ AD is perpendicular to BC.

47. *When* ABC *is an acute-angled triangle the angles of the triangle* DEF *in the previous figure are equal to the supplements of twice the angles of the original triangle* ABC, *and its sides are equally inclined to the sides of the triangle* ABC.

Since O, F, A, E lie on a circle,

∴ ∠OFE = ∠OAE = a right ∠ − ∠ACD. [III. 21.

So, since O, F, B, D lie on a circle,

∴ ∠OFD = ∠OBC = a right ∠ − ∠ECB.

$\therefore$ OFE and OFD are equal angles, and their sum

$$= \text{two right } \angle^s - 2\angle ACB.$$

that is, $\angle DFE =$ the supplement of twice the $\angle C$.

So the angles FDE, DEF are the supplements of twice the angles A and B.

Also since $\angle EFO = \angle DFO$, $\therefore$ $\angle EFA = \angle DFB$.

$\therefore$ FE, FD are equally inclined to AB.

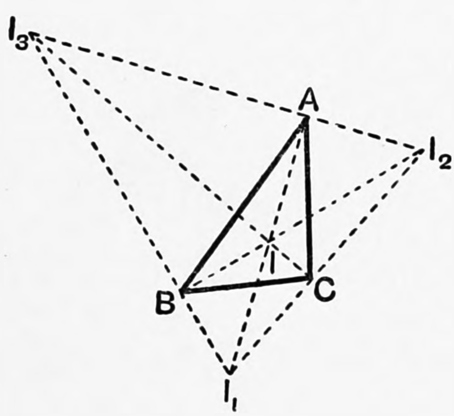
Similarly FE, ED are equally inclined to AC, and ED, DF are equally inclined to BC.

Definition. The triangle DEF is often called the **Pedal** Triangle of ABC.

48. *If* I *be the in-centre and* I_1, I_2, I_3 *the three e-centres of a triangle* ABC, *then*

(1) I *is the orthocentre of the triangle* $I_1I_2I_3$.

(2) ABC *is the pedal triangle of* $I_1I_2I_3$.



For AI bisects the $\angle BAC$, since AB, AC are tangents to a circle centre I. [III. 17, *Cor.* 2.

For a similar reason AI_1 bisects the same $\angle$.

$\therefore$ AII_1 is a straight line, and similarly BII_2, CII_3.

Also since BA produced and AC touch a circle centre I_2.

$\therefore$ AI_2 bisects the exterior angle of BAC. [III. 17, *Cor.* 2.

Also AI bisects $\angle BAC$.

$\therefore$ IAI_2 is a right angle. Similarly IAI_3 is a right $\angle$.

$\therefore$ I_2AI_3 is a straight line. Similarly I_3BI_1 and I_1CI_2 are straight lines.

$\therefore$ I_1IA, I_2IB, and I_3IC are the perpendiculars from I_1, I_2, I_3 on the sides of the $\triangle I_1I_2I_3$.

$\therefore$ I is the orthocentre and ABC the pedal $\triangle$ of the $\triangle I_1I_2I_3$.

EXERCISES.

****1.** The perpendicular from the middle point of the side of any triangle on the opposite side of its pedal triangle bisects the latter.

****2.** O is the orthocentre of a triangle ABC and the perpendicular AD on BC, when produced, meets the circumcircle in H; prove that DH = DO.

[For ∠DCH = ∠DAB (III. 21) = rt. ∠ − ABD = ∠BCF.

∴ △s OCD, HCD are equal in all respects; ∴ etc.]

****3.** If O is the orthocentre of the triangle ABC, then either of the four points O, A, B, C is the orthocentre of the triangle formed by the other three.

****4.** If ABC be a triangle obtuse-angled at A, and AD, BE, CF be the perpendiculars on the sides, then BE, CF bisect the exterior angles of the △DEF, and AD bisects the interior angle.

5. The radii from the circumcentre of a triangle to the angular points are respectively perpendicular to the straight lines joining the feet of the perpendiculars on the sides of the triangle from the opposite vertices.

****6.** With the letters of Art. 46, prove that

AO . OD = BO . OE = CO . OF, and that DB . DC = DO . DA, etc.

[Taking the left-hand figure, we have AO . AD = AF . AB, since F, B, D, O lie on a circle.

$$\therefore AO \cdot OD = AF \cdot AB - AO^2 = AF \cdot FB + AF^2 - AO^2 = AF \cdot FB - OF^2$$
$$= AF \cdot FB + BF^2 - BO^2 = BA \cdot BF - BO^2$$
$$= BO \cdot BE - BO^2 \text{ (since O, F, A, E lie on a circle)}$$
$$= BO \cdot OE.$$

This is more easily proved by Book VI., since EOA, DOB are similar △s].

****7.** In the second figure of Art. 46, if with centre O and radius, whose square is equal to either of the rectangles OA . OD, OB . OE, OC . OF, a circle be described, then each angular point of the triangle ABC is the polar with respect to this circle of the opposite side.

[This follows from the last Exercise and Art. 42. Such a circle is called the **Polar Circle** of the triangle; it is also called the **Self-Conjugate Circle**. If the △ be acute-angled, as in the first figure of Art. 46, there is no such circle. For its centre, if any, must, by Art. 42, lie on each of the three lines AD, BE, and CF, and must therefore be at O. But since in this case the point A and the line BC lie on **opposite** sides of O, it is impossible by Art. 42 that A should be the polar of BC.]

8. The circles described on the sides of a triangle as diameters cut the polar circle orthogonally.

[Taking the right-hand figure of Art. 46, the circle on AC as diameter passes through A and D, which are inverse points with respect to the polar circle, since OA . OD = sq. on radius of the polar circle. Then apply Art. 45.]

9. Find a point O within or without a triangle ABC such that the circumcircles of the triangles OAB, OBC, OCA are all equal.

10. Prove that the orthocentre and vertices of any triangle are the in- and escribed centres of the pedal triangle of the given triangle.

11. Given the pedal triangle DEF of a certain triangle ABC shew how to construct ABC.

12. If DEF is the pedal △ of the △ABC, the triangles DEC, EFA, FDB, ABC are all equiangular,
and their circumcircles are all of the same size.

13. In the figure of Art. 48 prove that
(1) the triangles BI_1C, CI_2A, AI_3B are equiangular.
(2) the four circles each of which passes through three of the four points I, I_1, I_2, I_3 are all equal.

14. The internal and external bisectors of the angle A of a triangle meet the base BC in E, E′ and the circumcircle in D and D′; prove that D is the orthocentre of the △EE′D′.

15. O is the orthocentre of a triangle ABC, and D, E, F the circumcentres of the triangles OAB, OBC, and OCA. Prove that ABC and DEF are equal in all respects.

16. ABCD is a quadrilateral inscribed in a circle; E and F are the orthocentres of the △ˢ ABC and ABD respectively; prove that CDFE is a ||ᵐ.

Shew also that the orthocentres of the four △ˢ ABC, BCD, CDA, DAB form a quadrilateral equal and similar to the given one.

[Use the Corollary to Art. 8 in this and the three following Exercises.]

17. Prove that the sum of the squares on the straight line joining the vertex of a triangle to the orthocentre, and on the opposite side = the sq. on the diameter of the circumcircle.

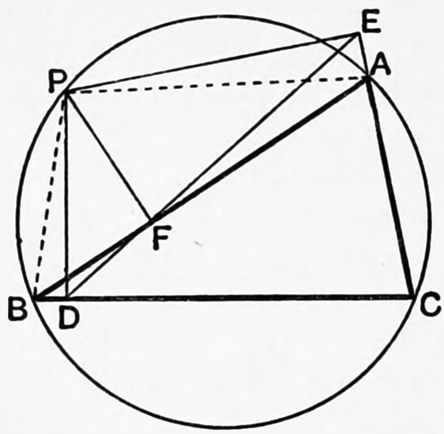
18. The orthocentre of a triangle is joined to the middle point of a side; prove that this joining line produced will meet the circumcircle in a point which is at the other end of the diameter through the angular point opposite to the bisected side.

19. A line MN of given length slides between two fixed lines OM, ON; prove that the orthocentre of the △OMN always lies on a certain circle.

20. A circular piece of paper is folded into any inscribed triangle; prove that the arcs of the folded segments all pass through a fixed point.

49. *If from any point in the circumference of the circle described round a triangle perpendiculars be drawn to the sides of the triangle, the three points of intersection are in the same straight line.*

Let ABC be a triangle, P any point on the circumcircle; from P draw PD, PE, PF perpendiculars to the sides BC, CA, AB respectively: D, E, F shall be in the same straight line.



[We will suppose that P is on the arc cut off by AB, on the opposite side from C, and that E is on CA produced through A; the demonstration will only have to be slightly modified for any other figure.]

A circle will go round PEAF; [III. 22 converse.

$\therefore$ $\angle$PFE $=$ $\angle$PAE [III. 21.

$=$ two rt. $\angle^s - \angle$PAC [I. 13.

$=$ $\angle$PBC. [III. 22.

Again, a circle will go round PFDB; [III. 21.

$\therefore$ $\angle$PFD $=$ two rt. $\angle^s - \angle$PBC. [III. 22 converse.

$\therefore$ $\angle^s$ PFD and PFE are together $=$ two right $\angle^s$.

$\therefore$ EF and FD are in the same straight line.

Conversely. If the feet of the perpendiculars PD, PE, PF drawn from any point P to the sides of a triangle ABC are collinear, then P lies on the circumcircle of ABC.

For $\angle$BPD $=$ $\angle$BFD (III. 21) $=$ $\angle$AFE $=$ $\angle$APE (III. 21).

$\therefore$ $\angle$PBD $=$ rt. $\angle - \angle$BPD $=$ rt. $\angle$ $-$ APE $=$ $\angle$PAE.

$\therefore$ $\angle$PBD $+$ $\angle$PAC $=$ 2 rt. $\angle^s$.

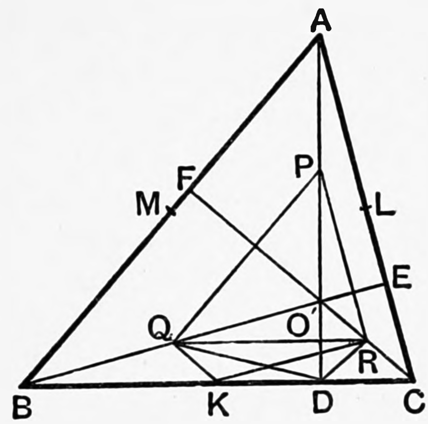
$\therefore$ P, A, B, C lie on a circle.

Definition. The above straight line DFE is sometimes called the **Pedal Line** of P, and sometimes it is called the **Simson Line** of P from the name of its supposed discoverer.

50. ABC *is a triangle, and* O′ *is its orthocentre; the circle which passes through the middle points of* O′A, O′B, O′C *will pass through the feet of*

the perpendiculars and through the middle points of the sides of the triangle.

Let P, Q, R be the middle points of OA, OB, OC respectively; let D be the foot of the perpendicular from A on BC, and K the middle point of BC.



Then O′BD is a right-angled triangle and Q is the middle point of the hypotenuse O′B;

$$\therefore\ QD = QO';$$

$$\therefore\ \angle QDO' = \angle QO'D.$$

Similarly, the $\angle RDO' = RO'D$;

$$\therefore\ \angle RDQ = RO'Q.$$

But $\angle^s$ CO′B, BAC together = two rt. $\angle^s$. [III. 22.

$\therefore$ $\angle^s$ RDQ, BAC together = two rt. $\angle^s$.

Also $\angle BAC = \angle QPR$, [Art. 1.

since QP, PR are $\|^l$ to BA, AC.

$\therefore$ $\angle^s$ RDQ, QPR = two rt. $\angle^s$.

$\therefore$ D is on the circle through P, Q, R.

Again, RK is parallel to O′B, and QK parallel to O′C; [Art. 1.

$$\therefore\ \angle QKR = \angle QO'C = \angle RDQ;$$

$\therefore$ K is also on the circumference of the circle.

Similarly, the two points in each of the other sides of the triangle ABC may be shewn to be on the circle.

51. The circle which is thus shewn to pass through these nine points is called the *Nine-point circle*: It has some curious properties, some of which we will now give.

52. *The radius of the nine-point circle is half of the radius of the circle described about the original triangle.*

For the $\triangle$ PQR has its sides respectively halves of the sides of the triangle ABC, so that the triangles are equiangular. Hence the radius

of the circle described round PQR is half of the radius of the circle described round ABC.

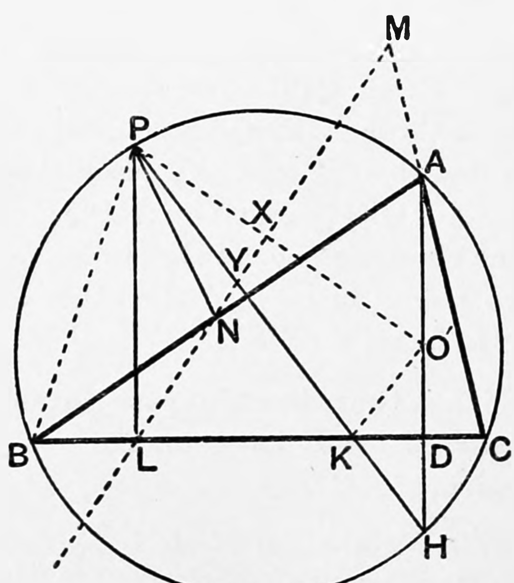
53. *If* O *be the centre of the circle described round the triangle* ABC, *the centre of the nine-point circle is the middle point of* OO′.

For OK is at right angles to BC, and therefore parallel to O′D. Hence the straight line which bisects KD at right angles must bisect OO′. And K and D are on the circumference of the nine-points circle, so that the straight line which bisects KD at right angles must pass through the centre of the nine-point circle. Similarly, from the other sides of the triangle ABC two other straight lines can be obtained, which pass through the centre of the nine-point circle and also bisect OO′. Hence the centre of the nine-point circle must coincide with the middle point of OO′.

Cor. The circumcentre O, the centroid G, the nine-point centre N, and the orthocentre O′ lie on a straight line; also G is a point of trisection [Art. 7], and N the point of bisection of OO′.

54. We may state that the nine-point circle of any triangle touches the inscribed circle and the escribed circles of the triangle: a demonstration of this theorem will be found in Dr. Todhunter's *Plane Trigonometry*, Chapter XXIV. For the history of the theorem see the *Nouvelles Annales de Mathématiques* for 1863, page 562.

55. *The line joining any point* P *on the circumcircle of any triangle* ABC *to the orthocentre* O′ *is bisected by the pedal line of* P *with respect to the triangle.*



Draw PL, PM, PN perpendicular to the sides of ABC; then LNM is the pedal line of P.

Draw AD perpendicular to BC and produce it to meet the circum-circle in H. Then, if O′ be the orthocentre, it lies on AD and O′D = DH. [Ex. 2, Art. 48.]

Let O′P meet LM in X. Then we have to prove that O′X = XP.

Join HP and let it meet BC in K. Join O′K.

Since PNB, PLB are both right angles, a circle goes through P, N, L, B;

$$\therefore \angle PLN = \angle PBN = \angle PBA = \angle PHA \qquad \text{[III. 21.}$$
$$= \angle HPL, \text{ since AH, PL are parallel.}$$
$$\therefore \angle YLP = \angle YPL.$$
$$\therefore \quad YP = YL,$$

and thus Y is the centre of the circumcircle of the $\triangle$PLK.

$$\therefore \; PY = YK = YL.$$

Again, since O′D = DH, and O′DK, HDK are right $\angle^s$,

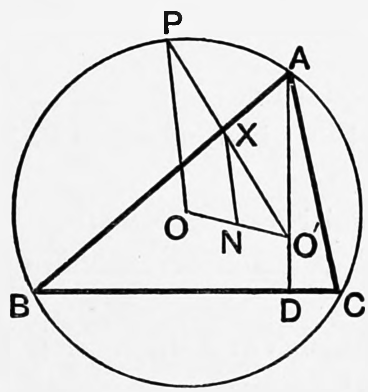
$$\therefore \angle O'KD = \angle HKD \qquad \text{[I. 4.}$$
$$= \angle YKL = \angle YLK, \text{ since } YK = YL. \quad (\textit{Proved.})$$

∴ O′K and LNM are parallel.

∴ LNM is a straight line drawn through the middle point Y of the side PK of the $\triangle$PKO′ parallel to its base KO′.

∴ LNM bisects PO′, *i.e.* PX = XO′. [Art. 1.

56. *The middle point of the straight line joining any point* P *on the circumcircle of any triangle to the orthocentre* O′ *lies on the nine-point circle of the triangle.*



Let O be the centre of the circumcircle. Bisect OO′ in N and join NX.

Then, by Art. 53, N is the centre of the nine-point circle.

Hence, since X and N are the middle points of O′P and O′O respectively, NX is parallel to, and is one-half of, OP.

$$\therefore \; NX = \text{one-half of the circum-radius}$$
$$= \text{the radius of the nine-point circle.} \qquad \text{[Art. 52.}$$

∴ X lies on the nine-point circle.

Hence the intersections of O′P with the pedal line of P always lies on the nine-point circle.

EXERCISES.

1. In the figure of Art. 50 prove that PQKL, PMKR are rectangles, and hence that the circle on PK as diameter goes through Q, L, M, R and also through D, E, F. [This is an easy method of proof of Art. 50.] Prove that PK, QL, RM meet at the centre of the nine-point circle.

2. If the perpendicular from any point P on the side BC of a △ ABC meet the circumcircle in Y, then AY is ||ˡ to the pedal line of P.

3. If through any point O on a circle three chords be drawn, and on each a circle be described, shew that these three circles, besides intersecting in O, meet in three points lying on a straight line.

[The three points are the feet of the ⊥ʳˢ from O upon the sides of the △ whose angular points are the other ends of the chords through O.]

4. *The pedal lines of a triangle with respect to two points on the circumcircle is equal to half the angle the two points subtend at the circumcentre.*

For, in the figure of Art. 55,

$\angle NLC = \angle LNB + B = \angle LPB + B = \text{rt.} \angle - \angle PBL + B$

$= \text{rt.} \angle - PBN = \text{rt.} \angle - \frac{1}{2} \angle POA$, where O is the circumcentre,

that is, inclination of pedal line of P to BC

$= \text{rt.} \angle - \frac{1}{2} \angle POA.$

So the inclination of the pedal line of any other point P′ to BC

$= \text{rt.} \angle - \frac{1}{2} P'OA.$

∴ angle between these pedal lines

$=$ the difference of these inclinations $= \frac{1}{2} \angle P'OA - \frac{1}{2} \angle POA$

$= \frac{1}{2} \angle P'OP.$

If P, P′ are at the extremities of a diameter of the circumcircle, then $\frac{1}{2} \angle P'OP =$ a right ∠, and ∴ the corresponding pedal lines are perpendicular.

5. *The pedal lines of the ends of a diameter of the circumcircle meet at right angles on the nine-point circle.*

For if the ends of the diameter be P, P′, and X, X′ be the middle points of O′P, O′P′, then XX′ bisects OO′ and thus passes through the nine-point centre. XX′ is thus a diameter of the nine-point circle. Also the pedal lines of P, P′ pass through X, X′, and they meet at a right ∠, by the last example; ∴ they meet on the nine-point circle. [III. 31.

6. P is any point on the circumcircle of the triangle ABC, and PL, drawn parallel to BC, meets the circle in Q; prove that AQ is perpendicular to the pedal line of P.

7. I is the in-centre and I_1, I_2, I_3 are the e-centres of a triangle ABC; prove that the straight lines II_1, II_2, II_3, I_2I_3, I_3I_1, I_1I_2 are all bisected by the circumcircle of the triangle ABC.

Prove also that the middle points of II_1 and I_2I_3 are at the ends of a diameter of this circle.

8. A, A′, B, C are four points on a circle. Prove that the pedal line of A′ with respect to the △ABC, of A with respect to the △A′BC, and the nine-point circles of these two △s meet in a point.

[This point is at the intersection of the diagonals of the parallelogram formed by A, A′ and the orthocentres of the two triangles.]

9. The nine-point circles of the triangles formed by four points, taken three at a time, meet in a point.

[Let A, B, C, D be the points, and P, Q, R, S, T, U the middle points of BC, CA, AB, BD, CD, AD. If O be the intersection of the nine-point circles of ABC and BCD, then $\angle POQ = \angle PRQ = \angle ACB$, and $\angle POT = \angle PST = \angle BCD$; $\therefore$ $\angle QOT = \angle ACD = \angle QUT$; $\therefore$ etc.]

10. If A, B, C, D be four points on a circle, the pedal lines of the triangles ABC, BCD, CDA, DAB with respect to D, A, B, C respectively meet in a point. [Use the two previous Exercises.]

11. Prove also that the centroids of the four triangles are concyclic.

12. A triangle inscribed in a given circle has its orthocentre at a fixed point. Prove that the middle points of its sides lie on a fixed circle.

13. Having given an angular point A of a triangle, its circumcentre, and the length of the base BC, the loci of its orthocentre and its nine-point centre are both circles.

14. Four concyclic points, taken three by three, determine four triangles whose nine-point centres are concyclic.

15. If the nine-point circle and one of the angular points of a triangle be given, the locus of the orthocentre is a circle.

16. O is the orthocentre of a triangle ABC; the nine-point circles of the triangles OAB, OBC, OCA, ABC all coincide.

17. D and E are points on the circumcircle of the triangle ABC, and their pedal lines meet in P; prove that the locus of P is a circle when A moves on the circumcircle and B, C, D, E are fixed.

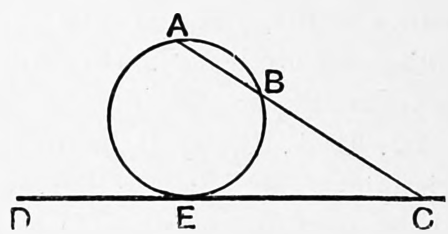
THE TANGENCIES OF CIRCLES.

57. A circle can be found to satisfy three conditions in general. Conversely, if three independent conditions be given, one circle, or at any rate only a finite number of circles can be found to satisfy them.

To describe a circle which shall pass through three given points not in the same straight line.

This is solved in Euclid IV. 5.

58. *To describe a circle which shall pass through two given points on the same side of a given straight line, and touch that straight line.*

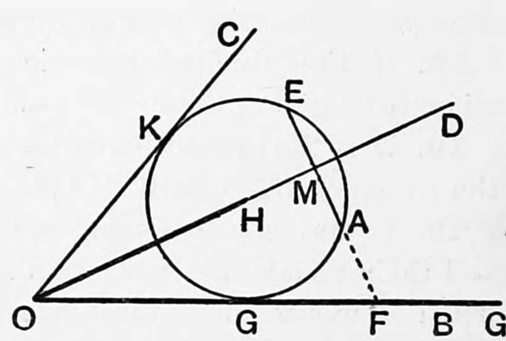


Let A and B be the given points; join AB and produce it to meet the given straight line at C. Make a square equal to the rectangle CA, CB (II. 14), and on the given straight line take CE equal to a side of this square. Describe a circle through A, B, E (IV. 5); this will be the circle required (III. 37).

Since E can be taken on either side of C, there are two solutions.

The construction fails if AB is parallel to the given straight line. In this case bisect AB at D, and draw DC at right angles to AB, meeting the given straight line at C. Then describe a circle through A, B, C.

59. *To describe a circle which shall pass through a given point and touch two given straight lines.*



Let A be the given point, and OB, OC the given straight lines. Bisect the angle BOC by the straight line OD. Draw AM perpendicular to OD and produce it to E, so that AM = ME.

Through A, E, draw a circle (by Art. 58) to touch the given straight line OB. This is done by producing EA to meet OB in F and taking $FG^2 = FA . FE$.

Since the circle passes through A and E, its centre lies on MO which bisects AE at right $\angle^s$. [III. 1.

Hence, if H be the centre, since it lies on the straight line OD which bisects the $\angle$BOC the perpendiculars HG, HK on OB, OC are equal,

and therefore the circle just drawn also touches OC. It is therefore the required circle.

There are two such circles; for if we take G′ on the other side of F from G, such that FG′=FG, a second circle can be drawn through A, E, and G′ which will also satisfy the required conditions.

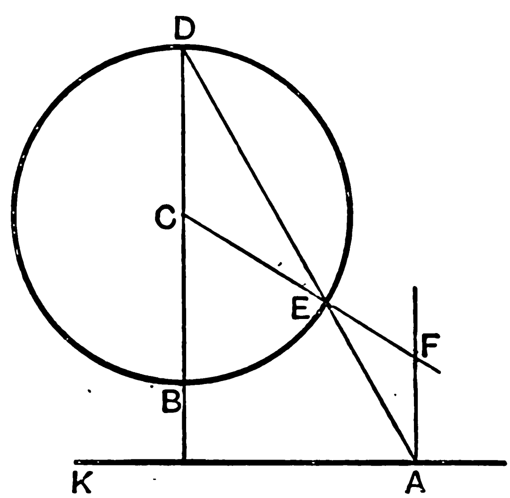
If A is on one of the given straight lines, draw from A a straight line at right angles to this given straight line; the point of intersection of this straight line with either of the two straight lines which bisect the angles made by the given straight lines may be taken for the centre of the required circle.

If the two given straight lines are parallel, instead of drawing a straight line BC to bisect the angle between them, we must draw it parallel to them, and equidistant from them.

60. *To describe a circle which shall touch three given straight lines, not more than two of which are parallel.*

Proceed as in Euclid IV. 4. If the given straight lines form a triangle, four circles can be described, namely, one as in Euclid, and three others, as on Page 181, each touching one side of the triangle and the other two sides produced. If two of the given straight lines are parallel, two circles can be described, namely, one on each side of the third given straight line.

61. *To describe a circle which shall touch a given circle, and touch a given straight line at a given point.*



Let A be the given point in the given straight line AK, and C be the centre of the given circle. Through C draw a straight line perpendicular to the given straight line to meet the circle in B and D, of which D is the more remote from the given straight line. Join AD, meeting the circle in E. From A draw a straight line at right angles to the given straight line, meeting CE produced at F.

Then F *shall be the centre of the required circle, and* FA *its radius.*

For the ∠AEF = the ∠CED; [I. 15.

and the ∠EAF = the ∠CDE; [I. 29.

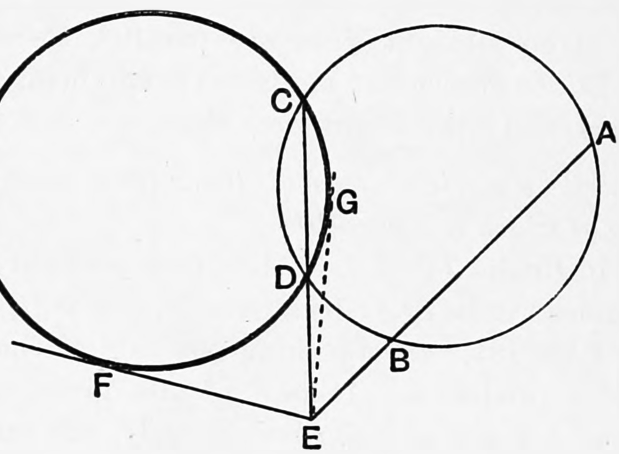
∴ the ∠AEF = the ∠EAF;

∴ AF = EF. [I. 6.

In a similar manner another solution may be obtained by joining AB. If the given straight line falls without the given circle, the circle obtained by the first solution touches the given circle externally, and the circle obtained by the second solution touches the given circle internally.

If the given straight line cuts the given circle, both the circles obtained touch the given circle externally.

62. *To describe a circle which shall pass through two given points and touch a given circle.*



Let A and B be the given points. Take any point C on the circumference of the given circle, and describe a circle through A, B, C. If this described circle touches the given circle, it is the required circle. But if not, let D be the other point of intersection of the two circles. Let AB and CD be produced to meet at E; from E draw a tangent EF to the given circle.

Then the circle through A, B, F shall be the required circle. See III. 35 and III. 37.

There are two solutions, because two tangents EF, EG can be drawn from E to the given circle.

If the straight line which bisects AB at right angles passes through the centre of the given circle, the construction fails, for AB and CD are parallel. In this case F must be determined by drawing a straight line parallel to AB so as to touch the given circle.

63. *To describe a circle which shall touch two given straight lines and a given circle.*

Draw two straight lines parallel to the given straight lines, at a distance from them equal to the radius of the given circle, and on the sides of them remote from the centre of the given circle. Describe a

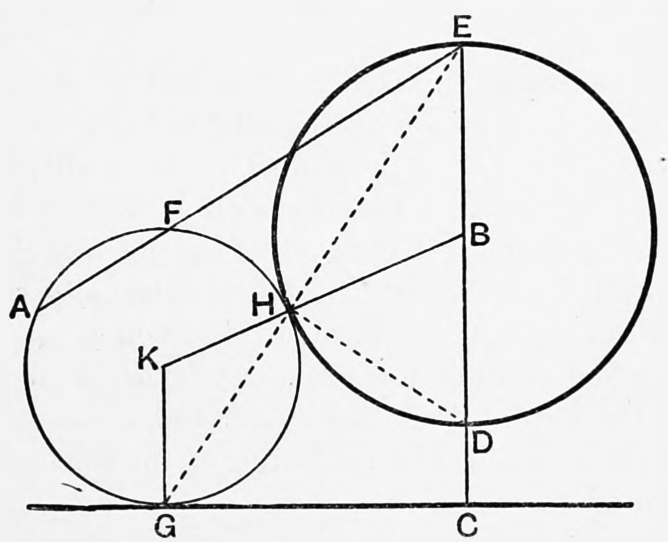
circle touching the straight lines thus drawn, and passing through the centre of the given circle (Art. 59). A circle having the same centre as the circle thus described, and a radius equal to the excess of its radius over that of the given circle, will be the required circle.

Two solutions will be obtained, because there are two solutions of the problem in Art. 59; the circles thus obtained touch the given circle externally.

We may obtain two circles which touch the given circle internally, by drawing the straight lines parallel to the given straight lines on the same sides of them as the centre of the given circle.

64. *To describe a circle which shall pass through a given point and touch a given straight line and a given circle.*

We will suppose the given point and the given straight line without the circle; other cases of the problem may be treated in a similar manner.



Let A be the given point, and B the centre of the given circle. From B draw a perpendicular to the given straight line, meeting it at C, and meeting the circumference of the given circle at D and E, so that D is between B and C.

Join EA and determine a point F in EA, produced if necessary, such that the rect. EA, EF = the rect. EC, ED; this can be done by describing a circle through A, C, D, which will meet EA at the required point. [III. 36, *Corollary*.

Describe a circle to pass through A and F and touch the given straight line (Art. 58); this shall be the required circle.

For, let the circle thus described touch the given straight line at G; join EG meeting the given circle at H, and join DH.

Since the $\angle^s$ GHD, GCD are right $\angle^s$, [III. 31, and *Construction*.

∴ H, G, C, D lie on a circle.

∴ rect. EC, ED = rect. EG, EH. [III. 36.

∴ H is on the described circle. [III. 36, *Corollary*.

Take K the centre of the described circle; join KG, KH, and BH. Then $\angle$KHG = $\angle$KGH [I. 5] = $\angle$HEB [I. 29] = $\angle$EHB [I. 5].

∴ KHB is a straight line;

∴ the described circle touches the given circle.

Two solutions will be obtained, because there are two solutions of the problem in Art. 58; the circles thus described touch the given circle externally.

By joining DA instead of EA we can obtain two solutions in which the circles described touch the given circle internally.

65. *To describe a circle which shall touch a given straight line and two given circles.*

Let A be the centre of the larger circle and B the centre of the smaller circle. Draw a straight line parallel to the given straight line, at a distance from it equal to the radius of the smaller circle, and on the side of it remote from A. Describe a circle with A as centre, and radius equal to the difference of the radii of the given circles. Describe a circle which shall pass through B, touch externally the circle just described, and also touch the straight line which has been drawn parallel to the given straight line [Art. 64]. Then a circle having the same centre as the second described circle, and a radius equal to the difference between its radius and the radius of the smaller given circle, will be the required circle.

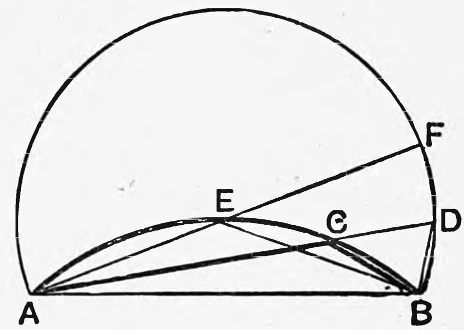
Two solutions will be obtained, because there are two solutions of the problem in Art. 64; the circles thus described touch the given circles externally.

We may obtain in a similar manner circles which touch the given circles internally, and also circles which touch one of the given circles internally and the other externally.

[For other cases of circles satisfying given conditions see Arts. 87, 88, 90, and 91.]

Constructions.

66. *Construct a triangle given its base, its vertical angle, and the sum of its sides equal to a given straight line.*



Let AB be the base; on it describe a segment of a circle containing an ∠ACB equal to the given vertical angle. Let C be a point on such that the sum of AC, CD is equal to the given sum.

Produce AC to D so that CD = CB, and join DB.

Then AD is equal to the given straight line.

Also the ∠ACB = the sum of the ∠s CDB and CBD [I. 32], that is, = twice the ∠CDB [I. 5].

∴ the ∠ADB is half of the given vertical ∠.

Hence we have the following solution. Describe on AB a segment of a circle containing an ∠ equal to the given ∠ and a second segment containing an ∠ equal to half the given ∠.

With A as centre, and a radius equal to the given straight line, describe a circle.

Join A with a point of intersection D of this circle and the second segment; this joining straight line will cut the circumference of the first segment at a point C which solves the problem.

The given straight line must exceed AB, and it must not exceed a certain straight line which we will now determine. Suppose the circumference of the first segment bisected at E; join AE, and produce it to meet the circumference of the second segment at F.

Then AE = EB [III. 28], and EB = EF,

since ∠s EFB, EBF = ∠AEB = ∠ACB

= twice ∠ADB = twice ∠AFB,

and ∴ ∠EBF = ∠EFB.

Thus EA, EB, EF are all equal;

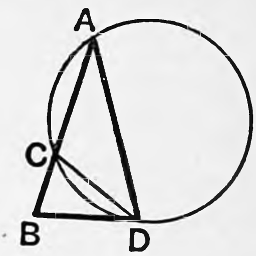
∴ E is the centre of the circle of which ADB is a segment. [III. 9.

Hence AF is the longest straight line which can be drawn from A to the circumference of the described segment; so that the given straight line must not exceed twice AE.

67. *To describe an isosceles triangle having each of the angles at the base double of the third angle.*

This problem is solved in IV. 10; we may suppose the solution to have been discovered by such an analysis as the following.

Suppose the triangle ABD such a △ as is required, so that each of the ∠s at B and D is double of the ∠ at A.



Bisect the ∠ at D by the straight line DC.

Then the ∠ ADC = the ∠ at A;

∴ CA = CD.

The ∠ CBD = the ∠ ADB, by hypothesis,

and the ∠ CDB = the ∠ at A;

∴ the third ∠ BCD = the third ∠ ABD. [I. 32.

∴ BD = CD. [I. 6.

∴ BD = AC.

Since ∠ BDC = the ∠ at A, BD is a tangent at D to the circle described round the △ ACD (*Note on* III. 32).

∴ the rect. AB, BC = the square on BD. [III. 36.

∴ the rect. AB, BC = the square on AC.

∴ AB is divided at C in the manner required in II. 11.

Hence the synthetical solution of the problem is evident.

EXERCISES.

Describe a triangle having given its base, its vertical angle, and

1. another side;

2. its altitude;

3. the point where the perpendicular from the vertical angle meets the base;

4. the difference of its sides;

5. the sum of the squares on its sides;

[By Page 145 the vertex lies on a known arc of a circle going through the ends of the base; also, by Page 109, Ex. 1, the vertex lies on a circle whose centre is the middle point of the base; the vertex is thus at the intersection of these two circles.]

6. the difference of the squares on its sides;

7. the point where the bisector of the vertical angle meets the base;

8. the length of the median through the vertex;

9. the length of the median through one end of the base;

10. the difference of the angles at the ends of its base.

Describe a circle

11. with a given radius to touch two given straight lines;

12. with a given radius to touch two given circles;

13. with a given radius to touch a given circle and a given straight line;

[Since the required circle is to touch the given circle its centre is somewhere on a circle whose centre is that of the given circle and whose radius = sum of radius of given circle and given radius; since it touches given straight line the centre is on a straight line parallel to the given one; $\therefore$ etc.]

14. with a given radius to pass through a given point and touch a given straight line;

15. with a given radius and centre in a given straight line to touch another given straight line;

16. through a given point to touch a given straight line at a given point;

17. to touch a given circle at a given point and also to touch a given straight line;

18. through a given point to touch a given circle at a given point;

19. to touch a given circle and also to touch two tangents to the circle;

20. to touch a given straight line at a given point and bisect the circumference of a given circle;

21. to pass through a given point and bisect the circumferences of two given circles;

22. Inscribe a circle in a given sector of a circle.

23. Find the centre of a circle cutting off equal chords from the sides.

24. With three given points as centre draw three circles which touch in pairs.

25. Find a point outside a circle such that the tangents from it to a circle meet at a given angle.

26. Draw a chord in a given circle equal to one given straight line and parallel to another.

Describe a triangle having given

27. the centres of its three escribed circles;

28. its in-centre and the centres of two escribed circles;

29. its pedal triangle;

30. its base, altitude, and circum-radius;

31. an $\angle$ A, the perp^r from A upon the opposite side, and the in-radius;

32. the vertical $\angle$, the perimeter, and the in-radius;

33. the vertical $\angle$, the perimeter, and the altitude;

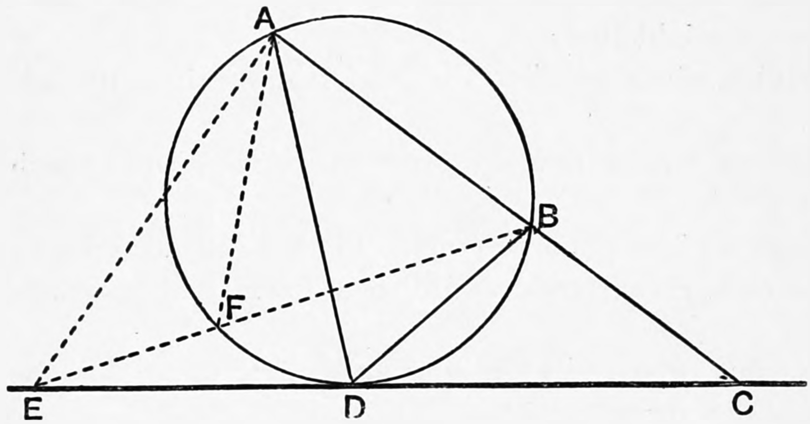
34. the base, one $\angle$ at the base, and the in-radius.

35. an angular point, the ortho-centre, and the circum-centre.

MAXIMA AND MINIMA [BOOKS III. AND IV.].

68. A *and* B *are two given points on the same side of a given straight line, and* AB *produced meets the given line at* C; *of all points in the given straight line on each side of* C, *it is required to determine that at which* AB *subtends the greatest angle.*

Describe a circle to pass through A and B, and to touch the given straight line on that side of C which is to be considered. Let D be the point of contact;
D shall be the required point.



For, take any other point E in CD on the same side of C as D; draw EA, EB; then one at least of these straight lines will cut the circumference ADB.

Suppose that BE cuts the circumference at F; join AF. Then the $\angle$AFB=the $\angle$ADB [III. 21]; and the $\angle$AFB>the $\angle$AEB [I. 16];

$\therefore$ the $\angle$ADB>the $\angle$AEB.

$\therefore$ the $\angle$ subtended by A, B at D is greater than the $\angle$ subtended at any other point of CD.

69. A *and* B *are two given points within a circle; and* AB *is drawn and produced both ways so as to divide the whole circumference into two arcs; it is required to determine the point in each of these arcs at which* AB *subtends the greatest angle.*

Describe a circle to pass through A and B and to touch the circumference considered [Art. 62]; the point of contact will be the required point. The demonstration is similar to that in the preceding proposition.

70. A *and* B *are two given points without a given circle; it is required to determine the points on the circumference of the given circle at which* AB *subtends the greatest and least angles.*

Suppose that neither AB nor AB produced cuts the given circle.

Describe two circles to pass through A and B, and to touch the

given circle [Art. 62]; then G the point of contact of the circle which touches the given circle externally, will be the point where the angle is greatest, and F, the point of contact of the circle which touches the given circle internally, will be the point where the angle is least. The demonstration is similar to that in Art. 68.

If AB cuts the given circle, both the circles obtained by Art. 62 touch the given circle internally; in this case the angle subtended by AB at a point of contact is less than the angle subtended at any other point of the circumference of the given circle which is on the same side of AB. Here the angle is greatest at the points where AB cuts the circle, and is there equal to two right angles.

If AB *produced* cuts the given circle, both the circles obtained by Art. 62 touch the given circle externally; in this case the angle subtended by AB at a point of contact is greater than the angle subtended at any other point of the circumference of the given circle which is on the same side of AB. Here the angle is least at the points where AB produced cuts the circle, and is there zero.

EXERCISES.

****1.** Given the base and vertical angle, construct it when its area is greatest, and show that the tangent at its vertex is then parallel to its base.

****2.** Of all triangles inscribed in a circle, the equilateral one is the greatest.

[For if any triangle is such that the tangent at one vertex A is not ||[l] to the base BC, we can by Ex. 1 obtain a greater △ by keeping BC fixed, and taking the vertex at a point where the tangent is ||[l] to the base; ∴ the greatest △ has the tangent at each angular point ||[l] to the opposite side; ∴ etc.]

3. The pedal triangle of a triangle ABC is the triangle of least perimeter with its vertices on the sides of ABC. [Use Art. 27.]

4. Of all quadrilaterals inscribed in a circle, the inscribed square has the greatest area and the greatest perimeter.

5. Of all chords drawn through a given point within a circle, the least is the one that is bisected at the given point; prove also that it cuts off the least area from the circle.

6. Given a circle and two tangents, prove that the tangent which is such that the portion of it intercepted between the two tangents is a minimum, is bisected at the point of contact. Prove also that it cuts off a maximum or a minimum triangle.

EXERCISES.

I. 1–31.

1. In the figure of I. 2 if the diameter of the smaller circle is the radius of the larger, shew where the given point and the vertex of the constructed triangle will be situated.

2. In the figure of I. 17 shew that ABC and ACB are together less than two right angles, by joining A to any point in BC.

3. Two right-angled triangles have their hypotenuses equal, and a side of one equal to a side of the other: shew that they are equal in all respects.

4. If a straight line be drawn bisecting one of the angles of a triangle to meet the opposite side, the straight lines drawn from the point of section parallel to the other sides, and terminated by these sides, will be equal.

5. The side BC of a triangle ABC is produced to a point D; the angle ACB is bisected by the straight line CE which meets AB at E. A straight line is drawn through E parallel to BC, meeting AC at F, and the straight line bisecting the exterior angle ACD at G. Shew that EF is equal to FG.

6. AB is the hypotenuse of a right-angled triangle ABC: find a point D in AB such that DB may be equal to the perpendicular from D on AC.

7. ABC is an isosceles triangle: find points D, E in the equal sides AB, AC such that BD, DE, EC may all be equal.

8. A straight line drawn at right angles to BC the base of an isosceles triangle ABC cuts the side AB at D and CA produced at E: shew that AED is an isosceles triangle.

I. 32.

9. Shew that any angle of a triangle is obtuse, right, or acute, according as it is greater than, equal to, or less than the other two angles of the triangle taken together.

10. Construct an isosceles triangle having the vertical angle four times each of the angles at the base.

11. Construct an isosceles triangle which shall have one-third of each angle at the base equal to half the vertical angle.

12. AB, AC are two straight lines given in position: it is required to find in them two points P and Q, such that, PQ being joined, AP and PQ may together be equal to a given straight line, and may contain an angle equal to a given angle.

13. Straight lines are drawn through the extremities of the base of an isosceles triangle, making angles with it, on the side remote from the vertex, each equal to one-third of one of the equal angles of the triangle and meeting the sides produced: shew that three of the triangles thus formed are isosceles.

14. AEB, CED are two straight lines intersecting at E; straight lines AC, DB are drawn forming two triangles ACE, BED; the angles ACE, DBE are bisected by the straight lines CF, BF, meeting at F. Shew that the angle CFB is equal to half the sum of the angles EAC, EDB.

15. From the angle A of a triangle ABC a perpendicular is drawn to the opposite side, meeting it, produced if necessary, at D; from the angle B a perpendicular is drawn to the opposite side, meeting it, produced if necessary, at E: shew that the straight lines which join D and E to the middle point of AB are equal.

16. From the angles at the base of a triangle perpendiculars are drawn to the opposite sides, produced if necessary: shew that the straight line joining the points of intersection will be bisected by a perpendicular drawn to it from the middle point of the base.

17. In the figure I. 1, if C and F be the points of intersection of the circles, and AB be produced to meet one of the circles at K, shew that CFK is an equilateral triangle.

18. The straight lines bisecting the angles at the base of an isosceles triangle meet the sides at D and E: shew that DE is parallel to the base.

19. AB, AC are two given straight lines, and P is a given point in the former: it is required to draw through P a straight line to meet AC at Q, so that the angle APQ may be three times the angle AQP.

20. From a given point it is required to draw to two parallel straight lines, two equal straight lines at right angles to each other.

21. In a right-angled triangle if one of the acute angles be double the other, the hypotenuse is double the shorter side.

22. On the sides AB, BC of a parallelogram ABCD equilateral triangles ABP, BCQ are described having their vertices remote from the parallelogram. Prove that PQD is an equilateral triangle.

23. In the figure of I. 1, if CA and CB on being produced meet the circles again in G and H, and the circles meet again in F, then GCH is an equilateral triangle and GF and FH are in the same straight line.

24. In an equiangular polygon each exterior angle is one-sixth of a right angle. How many sides has it?

25. In an equiangular polygon each interior angle is five-thirds of a right angle. How many sides has it?

26. What is the least number of triangles into which a polygon of n sides can be divided?

I. 33–34.

27. If a straight line which joins the extremities of two equal straight lines, not parallel, make the angles on the same side of it equal to each other, the straight line which joins the other extremities will be parallel to the first.

23. No two straight lines drawn from the extremities of the base of a triangle to the opposite sides can possibly bisect each other.

29. On the sides AB, BC, and CD of a parallelogram ABCD three equilateral triangles are described, that on BC towards the same parts as the parallelogram, and those on AB, CD towards the opposite parts: shew that the distances of the vertices of the triangles on AB, CD from that on BC are respectively equal to the two diagonals of the parallelogram.

30. If a six-sided plane rectilineal figure have its opposite sides equal and parallel, the three straight lines joining the opposite angles will meet at a point.

31. Inscribe a rhombus within a given parallelogram, so that one of the angular points of the rhombus may be at a given point in a side of the parallelogram.

32. ABCD is a parallelogram, and E, F, the middle points of AD and BC respectively; shew that BE and DF will trisect the diagonal AC.

33. Find a point D on the base BC of a triangle ABC such that, DE, DF be drawn parallel to AC, AB respectively, AE shall be equal to AF.

34. The feet of the perpendiculars drawn from A upon the internal and external bisectors of the angles B, C of the triangle ABC lie upon the straight line joining the middle points of AB and AC.

35. ABCD is a parallelogram, X any point on BC and Y any point on AX; prove that the triangles DXY and YBC are equal.

I. 35-45.

36. Construct a rhombus equal to a given parallelogram.

37. A straight line is drawn bisecting a parallelogram ABCD and meeting AD at E and BC at F: shew that the triangles EBF and CED are equal.

38. If a triangle is described having two of its sides equal to the diagonals of any quadrilateral, and the included angle equal to either of the angles between these diagonals, then the area of the triangle is equal to the area of the quadrilateral.

39. D, E are the middle points of the sides AB, AC of a triangle, and CD, BE intersect at F: shew that the triangle BFC is equal to the quadrilateral ADFE.

40. The sides AB, AC of a given triangle ABC are bisected at the points E, F; a perpendicular is drawn from A to the opposite side, meeting it at D. Shew that the angle FDE is equal to the angle BAC. Shew also that AFDE is half the triangle ABC.

41. Three parallelograms which are equal in all respects are placed with their equal bases in the same straight line and contiguous; the extremities of the base of the first are joined with the extremities of the side opposite to the base of the third, towards the same parts: shew that the portion of the new parallelogram cut off by the second is one half the area of any one of them.

42. ABCD is a parallelogram; from D draw any straight line DFG meeting BC at F and AB produced at G; draw AF and CG: shew that the triangles ABF, CFG are equal.

43. ABC is a given triangle: construct a triangle of equal area, having for its base a given straight line AD, coinciding in position with AB.

44. ABC is a given triangle: construct a triangle of equal area, having its vertex at a given point in BC and its base in the same straight line as AB.

45. ABCD is a given quadrilateral: construct another quadrilateral of equal area having AB for one side, and for another a straight line drawn through a given point in CD parallel to AB.

46. ABCD is a given quadrilateral: construct a triangle whose base shall be in the same straight line as AB, vertex at a given point P in CD, and area equal to that of the given quadrilateral.

47. ABC is a given triangle: construct a triangle of equal area, having its base in the same straight line as AB, and its vertex in a given straight line.

48. If through the point O within a parallelogram ABCD two straight lines are drawn parallel to the sides, and the parallelograms OB and OD are equal, the point O is in the diagonal AC.

I. 46–48.

49. A straight line is drawn intersecting the two sides of a right-angled triangle, and each of the acute angles is joined with the points where this straight line intersects the sides respectively opposite to them: shew that the squares on the joining straight lines are together equal to the square on the hypotenuse and the square on the straight line drawn parallel to it.

50. If any point P be joined to A, B, C, D, the angular points of a rectangle, the squares on PA and PC are together equal to the squares on PB and PD.

51. In a right-angled triangle if the square on one of the sides containing the right angle be three times the square on the other, and from the right angle two straight lines be drawn, one to bisect the opposite side, and the other perpendicular to that side, these straight lines divide the right angle into three equal parts.

52. On the hypotenuse BC, and the sides CA, AB of a right-angled triangle ABC, squares BDEC, AF, and AG are described: shew that the squares on DG and EF are together equal to five times the square on BC.

53. ABC is a triangle, right-angled at A, and the square on AB is three times that on AC; prove that the angle ACB is twice the angle ABC.

54. In the figure to I. 47, prove that the triangles DBF, ECK, AHG, ABC are equal.

I. 1–48.

55. From the centres A and B of two circles parallel radii AP, BQ are drawn; the straight line PQ meets the circumferences again at R and S: shew that AR is parallel to BS.

56. In the figure of I. 5, if the equal sides of the triangle be produced upwards through the vertex, instead of downwards through the base, a demonstration of I. 15 may be obtained without assuming any proposition beyond I. 5.

57. Shew that by superposition the first case of I. 26 may be immediately demonstrated, and also the second case, with the aid of I. 16.

58. A straight line is drawn, terminated by one of the sides of an isosceles triangle and by the other side produced, and bisected by the

base: shew that the straight lines thus intercepted between the vertex of the isosceles triangle and this straight line are together equal to the two equal sides of the triangle.

59. Of all parallelograms which can be formed with diagonals of given lengths the rhombus is the greatest.

60. If two equal straight lines intersect each other anywhere at right angles, the quadrilateral formed by joining their extremities is equal to half the square on either straight line.

61. Inscribe a parallelogram in a given triangle, in such a manner that its diagonals shall intersect at a given point within the triangle.

62. AB, AC are two given straight lines: it is required to find in AB a point P, such that if PQ be drawn perpendicular to AC, the sum of AP and AQ may be equal to a given straight line.

63. On a given straight line as base, construct a triangle, having given the difference of the sides and a point through which one of the sides is to pass.

64. Let one of the equal sides of an isosceles triangle be bisected at D, and let it also be doubled by being produced through the extremity of the base to E, then the distance of the other extremity of the base from E is double its distance from D.

65. Determine the locus of a point whose distance from one given point is double its distance from another given point.

66. A straight line AB is bisected at C, and on AC and CB as diagonals any two parallelograms ADCE and CFBG are described; let the parallelogram whose adjacent sides are CD and CF be completed, and also that whose adjacent sides are CE and CG: shew that the diagonals of these latter parallelograms are in the same straight line.

67. ABCD is a rectangle of which A, C are opposite angles; E is any point in BC and F is any point in CD: shew that twice the area of the triangle AEF, together with the rectangle BE, DF, is equal to the rectangle ABCD.

68. ABC, DBC are two triangles on the same base, and ABC has the side AB equal to the side AC; a circle passing through C and D has its centre E on CA, produced if necessary; a circle passing through B and D has its centre F on BA, produced if necessary: shew that the quadrilateral AEDF has the sum of two of its sides equal to the sum of the other two.

69. Two straight lines AB, AC are given in position: it is required to find in AB a point P, such that a perpendicular being drawn from it to AC, the straight line AP may exceed this perpendicular by a proposed length.

70. Shew that the opposite sides of any equiangular hexagon are parallel, and that any two sides which are adjacent are together equal to the two to which they are parallel.

71. ABC is a triangle in which C is a right angle: shew how to draw a straight line parallel to a given straight line, so as to be terminated by CA and CB, and bisected by AB.

72. ABC is an isosceles triangle having the angle at B four times either of the other angles; AB is produced to D so that BD is equal to twice AB, and CD is joined: shew that the triangles ACD and ABC are equiangular to one another.

73. Through a point K within a parallelogram ABCD straight lines are drawn parallel to the sides: shew that the difference of the parallelograms of which KA and KC are diagonals is equal to twice the triangle BKD.

74. Construct a right-angled triangle, having given one side and the difference between the other side and the hypotenuse.

75. BAC is a right-angled triangle; one straight line is drawn bisecting the right angle A, and another bisecting the base BC at right angles; these straight lines intersect at E: if D be the middle point of BC, shew that DE is equal to DA.

76. On AC the diagonal of a square ABCD, a rhombus AEFC is described of the same area as the square, and having its acute angle at A: if AF be joined, shew that the angle BAC is divided into three equal angles.

77. AB, AC are two fixed straight lines at right angles; D is any point in AB, and E is any point in AC; on DE as diagonal a half square is described with its vertex at G: shew that the locus of G is the straight line which bisects the angle BAC.

78. Shew that a square is greater than any other parallelogram of the same perimeter.

79. ABC is a triangle; AD is a third of AB, and AE is a third of AC; CD and BE intersect at F: shew that the triangle BFC is half the triangle BAC, and that the quadrilateral ADFE is equal to either of the triangles CFE or BDF.

80. ABC is a triangle, having the angle C a right angle; the angle A is bisected by a straight line which meets BC at D, and the angle B is bisected by a straight line which meets AC at E; AD and BE intersect at O: shew that the triangle AOB is half the quadrilateral ABDE.

81. Shew that a scalene triangle cannot be divided by a straight line into two parts which will coincide.

82. ABCD, ACED are parallelograms on equal bases BC, CE, and between the same parallels AD, BE; the straight lines BD and AE intersect at F: shew that BF is equal to twice DF.

83. Parallelograms AFGC, CBKH are described on AC, BC outside the triangle ABC; FG and KH meet at Z; ZC is joined, and through A and B straight lines AD and BE are drawn, both parallel to ZC, and meeting FG and KH at D and E respectively: shew that the figure ADEB is a parallelogram, and that it is equal to the sum of the parallelograms FC, CK.

84. If a quadrilateral have two of its sides parallel shew that the straight line drawn parallel to these sides through the intersection of the diagonals is bisected at that point.

85. Two triangles are on equal bases and between the same parallels: shew that the sides of the triangles intercept equal lengths of any straight line which is parallel to their bases.

86. In a right-angled triangle, right-angled at A, if the side AC be double of the side AB, the angle B is more than double of the angle C.

87. AHK is an equilateral triangle; ABCD is a rhombus, a side of which is equal to a side of the triangle, and the sides BC and CD of which pass through H and K respectively: shew that the angle A of the rhombus is ten-ninths of a right angle.

88. In the figure of I. 35 if two diagonals be drawn to the two parallelograms respectively, one from each extremity of the base, and the intersection of the diagonals be joined with the intersection of the sides (or sides produced) in the figure, shew that the joining straight line will bisect the base.

89. In the figure of I. 48 prove that AL, BK, CF meet in a point.

II. 1-14.

90. A straight line is divided into two parts; shew that if twice the rectangle of the parts is equal to the sum of the squares described on the parts, the straight line is bisected.

91. Construct a line the square on which shall be equal to the difference of two given squares.

92. If a straight line AB be bisected in C and produced to D so that the square on AD is three times the square on CD, and if CB be bisected in E, prove that the square on ED is three times the square on EB.

93. Divide a straight line into two parts so that the square on one part may equal twice the rectangle contained by the whole and the other part.

94. ABCDE is a straight line such that AB, BC, CD, DE are equal and O is an external point; prove that the difference of the squares on OA, OE is twice the difference of the squares on OB, OD.

95. In the figure of II. 11 if AB is produced to Q so that

$$BQ = BH \text{ then } AQ^2 = 5 \,.\, AH^2.$$

96. If a straight line be drawn through one of the angles of an equilateral triangle to meet the opposite side produced, so that the rectangle contained by the whole straight line thus produced and the part of it produced is equal to the square on the side of the triangle, shew that the square on the straight line so drawn will be double the square on a side of the triangle.

97. Divide a given straight line into two parts so that the rectangle contained by them may be equal to the square described on a given straight line which is less than half the straight line to be divided.

98. If the angle between two adjacent sides of a parallelogram increase, while their lengths do not alter, the diagonal through their point of intersection will diminish.

99. Produce one side of a given triangle so that the rectangle contained by this side and the produced part may be equal to the difference of the squares on the other two sides.

100. Construct a rectangle equal to a given square when the sum of two adjacent sides of the rectangle is equal to a given quantity.

101. Construct a rectangle equal to a given square when the difference of two adjacent sides of the rectangle is equal to a given quantity.

102. Two rectangles have equal areas and equal perimeters: shew that they are equal in all respects.

103. ABCD is a rectangle; P is a point such that the sum of PA and PC is equal to the sum of PB and PD: shew that the locus of P consists of the two straight lines through the centre of the rectangle parallel to its sides.

III. 1–15.

104. Two circles whose centres are A and B intersect at C; through C two chords DCE and FCG are drawn equally inclined to AB and terminated by the circles: shew that DE and FG are equal.

105. Through either of the points of intersection of two given circles draw the greatest possible straight line terminated both ways by the two circumferences.

106. If from any point in the diameter of a circle straight lines are drawn to the extremities of a parallel chord, the squares on these straight lines are together equal to the squares on the segments into which the diameter is divided.

107. A and B are two fixed points without a circle PQR; it is required to find a point P in the circumference, so that the sum of the squares described on AP and BP may be the least possible.

108. A circle is described on the radius of another circle as diameter, and two chords of the larger circle are drawn, one through the centre of the less at right angles to the common diameter, and the other at right angles to the first through the point where it cuts the less circle. Shew that these two chords have the segments of the one equal to the segments of the other, each to each.

109. O is the centre of a circle, P is any point in its circumference, PN a perpendicular on a fixed diameter: shew that the straight line which bisects the angle OPN always passes through one or the other of two fixed points.

110. Describe a circle which shall touch a given circle, have its centre in a given straight line, and pass through a given point in the given straight line.

III. 16-19.

111. A circle is drawn to touch a given circle and a given straight line. Shew that the points of contact are always in the same straight line with a fixed point in the circumference of the given circle.

112. Draw a straight line to touch one given circle so that the part of it contained by another given circle shall be equal to a given straight line not greater than the diameter of the latter circle.

113. Draw a straight line cutting two given circles so that the chords intercepted within the circles shall have given lengths.

114. ABD, ACE are two straight lines touching a circle at B and C, and if DE be joined DE is equal to BD and CE together: shew that DE touches the circle.

115. Two radii of a circle at right angles to each other when produced are cut by a straight line which touches the circle: shew that the tangents drawn from the points of section are parallel to each other.

116. If two circles can be described so that each touches the other and three of the sides of a quadrilateral figure, then the difference

between the sums of the opposite sides is double the common tangent drawn across the quadrilateral.

117. AB is the diameter and C the centre of a semicircle: shew that O, the centre of any circle inscribed in the semicircle, is equidistant from C and from the tangent to the semicircle parallel to AB.

118. A quadrilateral is bounded by the diameter of a circle, the tangents at its extremities, and a third tangent: shew that its area is equal to half that of the rectangle contained by the diameter and the side opposite to it.

119. If a quadrilateral, having two of its sides parallel, be described about a circle, a straight line drawn through the centre of the circle, parallel to either of the two parallel sides, and terminated by the other two sides, shall be equal to a fourth part of the perimeter of the figure.

120. A series of circles touch a fixed straight line at a fixed point: shew that the tangents at the points where they cut a parallel fixed straight line all touch a fixed circle.

121. Of all straight lines which can be drawn from two given points to meet on the convex circumference of a given circle, the sum of the two is least which make equal angles with the tangent at the point of concourse.

122. C is the centre of a given circle, CA a radius, B a point on a radius at right angles to CA; join AB and produce it to meet the circle again at D, and let the tangent at D meet CB produced at E: shew that BDE is an isosceles triangle.

123. Let the diameter BA of a circle be produced to P, so that AP equals the radius; through A draw the tangent AED, and from P draw PEC touching the circle at C and meeting the former tangent at E; join BC and produce it to meet AED at D: then will the triangle DEC be equilateral.

124. If two circles touch one another externally the square on the common tangent is equal to the rectangle contained by the diameters.

125. ABCD is a straight line, and circles are described on AB and CD as diameters, and a common tangent to the circles is drawn meeting them in E and F. Prove that the triangles AEB and CFD are equiangular.

126. AB is a diameter and AP a chord of a circle; AQ is a chord bisecting the angle BAP: prove that the tangent at Q is perpendicular to AP.

127. The difference of the squares on the tangents drawn from any point to two concentric circles is equal to the square on the tangent drawn to the inner circle from any point on the outer circle.

III. 20-22.

128. Divide a circle into two parts so that the angle contained in one segment shall be equal to twice the angle contained in the other.

129. Divide a circle into two parts so that the angle contained in one segment shall be equal to five times the angle contained in the other.

130. If the angle contained by any side of a quadrilateral and the adjacent side produced be equal to the opposite angle of the quadrilateral, shew that any side of the quadrilateral will subtend equal angles at the opposite angles of the quadrilatera

131. If a quadrilateral be inscribed in a circle, and a straight line be drawn making equal angles with one pair of opposite sides, it will make equal angles with the other pair.

132. A quadrilateral can have one circle inscribed in it and another circumscribed about it: shew that the straight lines joining the opposite points of contact of the inscribed circle are perpendicular to each other.

133. A and B are two points on a circle centre C. An arc of a circle is drawn through A, C, and B, and a straight line APQ is drawn to cut the two circles in P and Q. Prove that PB and PQ are equal.

134. On the sides BC, CA, AB of a triangle any points α, β, γ are taken. Prove that the centres of the circles circumscribing the triangles $\beta A\gamma$, $\gamma B\alpha$, $\alpha C\beta$ are the angular points of a triangle equiangular with ABC.

135. A and B are fixed points on a circle and N any other point on AB. Two circles are drawn, each passing through N and touching the first circle in A and B respectively. Prove that the point of intersection lies on a fixed circl

136. A, B, C, D are four points on the circumference of a circle and the arcs AB, BC, CD, DA are bisected in E, F, G, H respectively; prove that EG, FH are perpendicular.

137. Circles are described upon the sides of any quadrilateral inscribed in a circle as diameters; prove that they intersect again in four points lying on a circle and that these four points form a quadrilateral equiangular with the given one.

138. Circles are described on the sides of any quadrilateral ABCD as diameters, and the four points of intersection of consecutive circles form a quadrilateral PQRS. Prove that the angles of PQRS and ABCD are equal or supplementary.

III. 23–30.

139. Through a point C in the circumference of a circle two straight lines ACB, DCE are drawn cutting the circle at B and E: shew that the straight line which bisects the angles ACE, DCB meets the circle at a point equidistant from B and E.

140. AB is a diameter of a circle, and D is a given point on the circumference, such that the arc DB is less than half the arc DA: draw a chord DE on one side of AB so that the arc EA may be three times the arc BD.

141. From A and B, two of the angular points of a triangle ABC, straight lines are drawn so as to meet the opposite sides at P and Q in given equal angles: shew that the straight line joining P and Q will be of the same length in all triangles on the same base AB, and having vertical angles equal to C.

142. OA, OB, OC are three chords of a circle; the angle AOB is equal to the angle BOC, and OA is nearer to the centre than OB. From B a perpendicular is drawn on OA, meeting it at P, and a perpendicular on OC produced, meeting it at Q: shew that AP is equal to CQ.

143. AB is a given finite straight line; through A two indefinite straight lines are drawn equally inclined to AB; any circle passing through A and B meets these straight lines at L and M. Shew that if AB be between AL and AM the sum of AL and AM is constant; if AB be not between AL and AM, the difference of AL and AM is constant.

144. AOB and COD are diameters of a circle at right angles to each other; E is a point in the arc AC, and EFG is a chord meeting COD at F, and drawn in such a direction that EF is equal to the radius. Shew that the arc BG is equal to three times the arc AE.

145. If two circles touch each other internally, any chord of the greater circle which touches the less shall be divided at the point of its contact into segments which subtend equal angles at the point of contact of the two circles.

146. ABCD is a semicircle whose diameter is AD; the chord BC produced meets AD produced in E; if CE is equal to the radius, prove that the arc AB is equal to three times the arc CD.

147. If two equal circles be drawn cutting each other in A and B, and if from A a chord is drawn cutting them in C and D, prove that the part CD between the circumferences is bisected by the circle on AB as diameter.

III. 31.

148. The greatest rectangle which can be inscribed in a circle is a square.

149. The hypotenuse AB of a right-angled triangle ABC is bisected at D, and EDF is drawn at right angles to AB, and DE and DF are cut off each equal to DA; CE and CF are joined: shew that the last two straight lines will bisect the angle C and its supplement respectively.

150. Describe a circle touching a given straight line at a given point, such that the tangents drawn to it from two given points in the straight line may be parallel.

151. Describe a circle with a given radius touching a given straight line, such that the tangents drawn to it from two given points in the straight line may be parallel.

152. AD is a diameter of a circle; B and C are points on the circumference on the same side of AD; a perpendicular from D on BC produced through C meets it at E: shew that the square on AD is greater than the sum of the squares on AB, BC, CD by twice the rectangle BC, CE.

153. AB is the diameter of a semicircle, P is a point on the circumference, PM is perpendicular to AB; on AM, BM as diameters two semicircles are described, and AP, BP meet these latter circumferences at Q, R: shew that QR will be a common tangent to them.

154. AB, AC are two straight lines, B and C are given points in the same; BD is drawn perpendicular to AC, and DE perpendicular to AB; in like manner CF is drawn perpendicular to AB, and FG to AC. Shew that EG is parallel to BC.

155. Two circles intersect at the points A and B, from which are drawn chords to a point C in one of the circumferences, and these chords, produced if necessary, cut the other circumference at D and E; shew that the straight line DE cuts at right angles that diameter of the circle ABC which passes through C.

156. If squares be described on the sides and hypotenuse of a right-angled triangle, the straight line joining the intersection of the diagonals of the latter square with the right angle is perpendicular to the straight line joining the intersections of the diagonals of the two former.

157. C is the centre of a given circle, CA a straight line less than the radius; find the point of the circumference at which CA subtends the greatest angle.

158. AB is the diameter of a semicircle, D and E are any two

points in its circumference. Shew that if the chords joining A and B with D and E each way intersect at F and G, then FG produced is at right angles to AB.

159. Two equal circles touch one another externally, and through the point of contact chords are drawn, one to each circle, at right angles to each other: shew that the straight line joining the other extremities of these chords is equal and parallel to the straight line joining the centres of the circles.

160. A circle is described on the shorter diagonal of a rhombus as a diameter, and cuts the sides; and the points of intersection are joined crosswise with the extremities of that diagonal: shew that the parallelogram thus formed is a rhombus with angles equal to those of the first.

161. Through a fixed point O a straight line is drawn, cutting a fixed circle in P and Q, and upon OP, OQ as chords are described circles touching the fixed circles in P and Q. Prove that the two circles so described will intersect on another fixed circle.

162. Two segments of circles are on the same base AB, and P is any point on one of them; the straight lines APD, BPC are drawn meeting the circumferences of the other segment in D and C; AC and BD meet in Q. Prove that the angle AQB is constant.

163. Any point P is taken on the given segment of a circle described on a line AB, and AG and BH are let fall upon BP and AP respectively; prove that GH touches a fixed circle.

164. A straight line AD is trisected at BC. On AB, BD as diameters circles are described; shew how to draw a straight line through A so that the parts of it intercepted by the two circles may be equal in length.

165. Through a fixed point A on a given circle a line is drawn cutting the circle at P, and AP is produced to Q, so that PQ is constant. Prove that the straight line through Q, perpendicular to AQ, touches a fixed circle.

166. AB is the diameter of a circle, and AC, AD two chords which when produced meet the tangent at B in E and F. Prove that a circle will pass through C, D, E, F.

III. 32–34.

167. If two circles touch each other any straight line drawn through the point of contact will cut off similar segments.

168. AB is any chord, and AD is a tangent to a circle at A. DPQ is any straight line parallel to AB, meeting the circumference at P and Q. Shew that the triangle PAD is equiangular to the triangle QAB.

169. Two circles ABDH, ABG, intersect each other at the points A, B; from B a straight line BD is drawn in the one to touch the other; and from A any chord whatever is drawn cutting the circles at G and H: shew that BG is parallel to DH.

170. Two circles intersect at A and B. At A the tangents AC, AD are drawn to each circle and terminated by the circumference of the other. If CB, BD be joined, shew that AB or AB produced, if necessary, bisects the angle CBD.

171. AB is any chord of a circle, P any point on the circumference of the circle; PM is a perpendicular on AB and is produced to meet the circle at Q, and AN is drawn perpendicular to the tangent at P: shew that the triangle NAM is equiangular to the triangle PAQ.

172. Two diameters AOB, COD of a circle are at right angles to each other; P is a point in the circumference; the tangent at P meets COD produced at Q, and AP, BP meet the same line at R, S respectively: shew that RQ is equal to SQ.

173. From a given point A without a circle, whose centre is O, draw a straight line cutting the circle at the points B and C, so that the area BOC may be the greatest possible.

174. Find a point within a triangle at which the three sides subtend equal angles.

175. ABC is a triangle inscribed in a circle, and AP, BQ are drawn parallel to BC, CA respectively to meet the circle in P and Q; prove that the chord PQ is parallel to the tangent to the circle at C.

176. Two circles PQA, PQB cut at P and Q. The tangent at P to the circle PQB meets the circle PQA at A, and the tangent at P to PQA meets PQB at B. Prove that the angles PQA, PQB are equal.

177. PMT is a tangent to the circle APC at the point P; CNAT is a diameter to which PN is drawn perpendicular and AM is perpendicular to PT; prove that AM, AN are equal.

178. In a right-angled triangle, if a semicircle be described on one of the sides, the tangent at the point where it cuts the hypotenuse bisects the other side.

179. The diagonals of a cyclic quadrilateral ABCD intersect in E; prove that the tangent at E to the circle about the triangle ABE is parallel to CD.

III. 35-37.

180. Two circles ABCD, EBCF, having the common tangents AE and DF, cut one another at B and C, and the chord BC is produced to cut the tangents at G and H: shew that the square on GH exceeds the square on AE or DF by the square on BC.

181. ABC is a right-angled triangle; from any point D in the hypotenuse BC a straight line is drawn at right angles to BC, meeting CA at E and BA produced at F: shew that the square on DE is equal to the difference of the rectangles BD, DC and AE, EC; and that the square on DF is equal to the sum of the rectangles BD, DC and AF, FB.

182. It is required to find a point in the straight line which touches a circle at the end of a given diameter, such that when a straight line is drawn from this point to the other extremity of the diameter, the rectangle contained by the part of it without the circle and the part within the circle may be equal to a given square not greater than that on the diameter.

III. 1-37.

183. AD, BE are perpendiculars from the angles A and B of a triangle on the opposite sides; BF is perpendicular to ED or DE produced: shew that the angle FBD is equal to the angle EBA.

184. If ABC be a triangle, and BE, CF the perpendiculars from the angles on the opposite sides, and K the middle point of the third side, shew that the angles FEK, EFK are each equal to A.

185. AB is a diameter of a circle; AC and AD are two chords meeting the tangent at B at E and F respectively: shew that the angles FCE and FDE are equal.

186. Two circles cut one another at a point A: it is required to draw through A a straight line so that the extreme length of it intercepted by the two circles may be equal to that of a given straight line.

187. Draw from a given point in the circumference of a circle, a chord which shall be bisected by its point of intersection with a given chord of the circle.

188. When an equilateral polygon is described about a circle it must necessarily be equiangular if the number of sides be odd, but not otherwise.

189. If any number of triangles on the same base BC, and on the same side of it have their vertical angles equal, and perpendiculars, intersecting at D, be drawn from B and C on the opposite sides, find the locus of D; and shew that all the straight lines which bisect the angle BDC pass through the same point.

190. Let O and C be any fixed points on the circumference of a circle, and OA any chord; then if AC be joined and produced to B, so that OB is equal to OA, the locus of B is an equal circle.

191. From any point P in the diagonal BD of a parallelogram ABCD, straight lines PE, PF, PG, PH are drawn perpendicular to the sides AB, BC, CD, DA: shew that EF is parallel to GH.

192. Through any fixed point of a chord of a circle other chords are drawn: shew that the straight lines from the middle point of the first chord to the middle points of the others will meet them all at the same angle.

193. ABC is a straight line, divided at any point B into two parts; ADB and CDB are similar segments of circles, having the common chord BD; CD and AD are produced to meet the circumferences at F and E respectively, and AF, CE, BF, BE are joined: shew that ABF and CBE are isosceles triangles, equiangular to one another.

194. A is a given point: it is required to draw from A two straight lines which shall contain a given angle and intercept on a given straight line a part of given length.

195. A and B are the centres of two circles which touch internally at C, and also touch a third circle, whose centre is D, externally and internally respectively at E and F: shew that the angle ADB is double of the angle ECF.

196. C is the centre of a circle, and CP is the perpendicular on a chord APB: shew that the sum of CP and AP is greatest when CP is equal to AP.

197. AB, BC, CD are three adjacent sides of any polygon inscribed in a circle; the arcs AB, BC, CD are bisected at L, M, N; and LM cuts BA, BC respectively at P and Q: shew that BPQ is an isosceles triangle, and that the angles ABC, BCD are together double of the angle LMN.

198. In the circumference of a given circle determine a point so situated that if chords be drawn to it from the extremities of a given chord of the circle their difference shall be equal to a given straight line less than the given chord.

199. Construct a triangle, having given the sum of the sides, the difference of the segments of the base made by the perpendicular from the vertex, and the difference of the base angles.

200. AKL is a fixed straight line cutting a given circle at K and L; APQ, ARS are two other straight lines making equal angles with AKL, and cutting the circle at P, Q and R, S: shew that whatever be the position of APQ and ARS, the straight line joining the middle points of PQ and RS always remains parallel to itself.

201. If about a quadrilateral another quadrilateral can be described such that every two of its adjacent sides are equally inclined to that

side of the former quadrilateral which meets them both, then a circle may be described about the former quadrilateral.

202. Two circles touch one another internally at the point A: it is required to draw from A a straight line such that the part of it between the circles may be equal to a given straight line, which is not greater than the difference between the diameters of the circles.

203. ABCD is a parallelogram; AE is at right angles to AB, and CE is at right angles to CB: shew that ED, if produced, will cut AC at right angles.

204. The two angles at the base of a triangle are bisected by two straight lines on which perpendiculars are drawn from the vertex: shew that the straight line which passes through the feet of these perpendiculars will be parallel to the base and will bisect the sides.

205. In a given circle inscribe a rectangle equal to a given rectilineal figure.

206. In an acute-angled triangle ABC perpendiculars AD, BE are let fall on BC, CA respectively; circles described on AC, BC as diameters meet BE, AD respectively at F, G and H, K: shew that F, G, H, K lie on the circumference of a circle.

207. Two diameters in a circle are at right angles; from their extremities four parallel straight lines are drawn: shew that they divide the circumference into four equal parts.

208. E is the middle point of a semicircle arc AEB, and CDE is any chord cutting the diameter at D, and the circle at C: shew that the square on CE is twice the quadrilateral AEBC.

209. AB is a fixed chord of a circle, AC is a moveable chord of the same circle; a parallelogram is described of which AB and AC are adjacent sides: find the locus of the middle points of the diagonals of the parallelogram.

210. AB is a fixed chord of a circle, AC is a moveable chord of the same circle; a parallelogram is described of which AB and AC are adjacent sides: determine the greatest possible length of the diagonal drawn through A.

211. If two equal circles be placed at such a distance apart that the tangent drawn to either of them from the centre of the other is equal to a diameter, shew that they will have a common tangent equal to the radius.

212. Find a point in a given circle from which if two tangents be drawn to an equal circle, given in position, the chord joining the points of contact is equal to the chord of the first circle formed by joining the points of intersection of the two tangents produced; and determine the limit to the possibility of the problem.

213. AB is a diameter of a circle, and AF is any chord; C is any point in AB, and through C a straight line is drawn at right angles to AB, meeting AF, produced if necessary at G, and meeting the circumference at D: shew that the rectangle FA, AG, and the rectangle BA, AC, and the square on AD are all equal.

214. A, B, C are three given points in the circumference of a given circle: find a point P such that if AP, BP, CP meet the circumference at D, E, F respectively, the arcs DE, EF may be equal to given arcs.

215. Find the point in the circumference of a given circle, the sum of whose distances from two given straight lines at right angles to each other, which do not cut the circle, is the greatest or least possible.

216. On the sides of a triangle segments of a circle are described *internally*, each containing an angle equal to the excess of two right angles above the opposite angle of the triangle: shew that the radii of the circles are equal, that the circles all pass through one point, and that their chords of intersection are respectively perpendicular to the opposite sides of the triangle.

IV. 1-4.

217. In IV. 3 shew that the straight lines drawn through A and B to touch the circle will meet.

218. In IV. 4 shew that the straight lines which bisect the angles B and C will meet.

219. In IV. 4 shew that the straight line DA will bisect the angle at A.

220. Two opposite sides of a quadrilateral are together equal to the other two, and each of the angles is less than two right angles. Shew that a circle can be inscribed in the quadrilateral.

221. Two circles HPL, KPM, that touch each other externally, have the common tangents HK, LM; HL and KM being joined, shew that a circle may be inscribed in the quadrilateral HKML.

222. Given the base, the vertical angle, and the radius of the inscribed circle of a triangle, construct it.

IV. 5-9.

223. Shew that if the straight line joining the centres of the inscribed and circumscribed circles of a triangle passes through one of its angular points, the triangle is isosceles.

224. The common chord of two circles is produced to any point P; PA touches one of the circles at A, PBC is any chord of the other,

Shew that the circle which passes through A, B, and C touches the circle to which PA is a tangent.

225. Describe a circle which shall pass through two given points and cut off from a given straight line a chord of given length.

226. Describe a circle which shall have its centre in a given straight line, and cut off from two given straight lines chords of equal given length.

227. Describe a circle which shall pass through two given points, so that the tangent drawn to it from another given point may be of a given length.

228. C is the centre of a circle; CA, CB are two radii at right angles; from B any chord BP is drawn cutting CA at N; prove that the circle described about the triangle ANP will be touched by BA.

229. The angle ACB of any triangle is bisected, and the base AB is bisected at right angles, by straight lines which intersect at D: shew that the angles ACB, ADB are together equal to two right angles.

230. ACDB is a semicircle, AB being the diameter, and the two chords AD, BC intersect at E: shew that if a circle be described round CDE it will cut the former at right angles.

231. A circle is described round the triangle ABC; the tangent at C meets AB produced at D; the circle whose centre is D and radius DC cuts AB at E: shew that EC bisects the angle ACB.

232. AB, AC are two straight lines given in position; BC is a straight line of given length; D, E are the middle points of AB, AC; DF, EF are drawn at right angles to AB, AC respectively. Shew that AF will be constant for all positions of BC.

233. A circle is described about an isosceles triangle ABC in which AB is equal to AC; from A a straight line is drawn meeting the base at D and the circle at E: shew that the circle which passes through B, D, and E, touches AB.

234. AC is a chord of a given circle; B and D are two given points in the chord, both within or both without the circle: if a circle be described to pass through B and D, and touch the given circle, shew that AB and CD subtend equal angles at the point of contact.

235. A and B are two points within a circle: find the point P in the circumference such that if PAH, PBK be drawn meeting the circle at H and K, the chord HK shall be the greatest possible.

236. The centre of a given circle is equidistant from two given straight lines: describe another circle which shall touch these two straight lines and shall cut off from the given circle a segment containing an angle equal to a given angle.

IV. 10–16.

237. On a given straight line as base describe an isosceles triangle having the third angle treble of each of the angles at the base.

238. If A be the vertex and BD the base of the constructed triangle in IV. 10, D being one of the two points of intersection of the two circles employed in the construction, and E the other, and AE be drawn meeting BD produced at G, shew that GAB is another isosceles triangle of the same kind.

239. In the figure of IV. 10 if the two equal chords of the smaller circle be produced to cut the larger, and these points of section be joined, another triangle will be formed having the property equired by the proposition.

240. In the figure of IV. 10 if AF be the diameter of the smaller circle, DF is equal to a radius of the circle which circumscribes the triangle BCD.

241. Shew that each of the triangles made by joining the extremities of adjoining sides of a regular pentagon is less than a third and greater than a fourth of the whole area of the pentagon.

242. Shew how to derive a regular hexagon from an equilateral triangle inscribed in a circle, and from the construction shew that the side of the hexagon equals the radius of the circle, and that the hexagon is double of the triangle.

243. In a given circle inscribe a triangle whose angles are as the numbers 2, 5, 8.

244. If ABCDEF is a regular hexagon, and AC, BD, CE, DF, EA, FB be joined, another regular hexagon is formed whose area is one third of that of the former.

IV. 1–16.

245. The points of contact of the inscribed circle of a triangle are joined; and from the angular points of the triangle so formed perpendiculars are drawn to the opposite sides: shew that the triangle of which the feet of these perpendiculars are the angular points has its sides parallel to the sides of the original triangle.

246. Construct a triangle having given an angle and the radii of the inscribed and circumscribed circles.

247. ABCDE is a regular pentagon; join AC and BD intersecting at O: shew that AO is equal to DO, and that the rectangle AC CO is equal to the square on BC.

248. A straight line PQ of given length moves so that its ends are always on two fixed straight lines CP, CQ; straight lines from P and Q at right angles to CP and CQ respectively intersect at R; perpendiculars from P and Q on CQ and CP respectively intersect at S: shew that the loci of R and S are circles having their common centre at C.

249. Right-angled triangles are described on the same hypotenuse: shew that the locus of the centres of the inscribed circles is a quarter of the circumference of a circle of which the common hypotenuse is a chord.

250. On a given straight line AB any triangle ACB is described; the sides AC, BC are bisected and straight lines drawn at right angles to them through the points of bisection to intersect at a point D; find the locus of D.

251. Construct a triangle, having given its base, one of the angles at the base, and the distance between the centre of the inscribed circle and the centre of the circle touching the base and the sides produced.

252. Within a given circle inscribe three equal circles, touching one another and the given circle.

253. If the radius of a circle be cut as in II. 11, the square on its greater segment, together with the square on the radius is equal to the square on the side of a regular pentagon inscribed in the circle.

254. From the vertex of a triangle draw a straight line to the base so that the square on the straight line may be equal to the rectangle contained by the segments of the base.

255. The perpendiculars from the angles A and B of a triangle ABC on the opposite sides meet at D; the circles described round ADC and DBC cut AB or AB produced at the points E and F: shew that AE is equal to BF.

256. Four circles are described so that each may touch internally three of the sides of a quadrilateral: shew that a circle may be described so as to pass through the centres of the four circles.

257. A circle is described round the triangle ABC, and from any point P of its circumference perpendiculars are drawn to BC, CA, AB, which meet the circle again at D, E, F: shew that the triangles ABC and DEF are equal in all respects, and that the straight lines AD, BE, CF are parallel.

258. With any point in the circumference of a given circle as centre, describe another circle, cutting the former at A and B; from B draw in the described circle a chord BD equal to its radius, and join AD, cutting the given circle at Q: shew that QD is equal to the radius of the given circle.

259. A point is taken without a square, such that straight lines being drawn to the angular points of the square, the angle contained by the two extreme straight lines is divided into three equal parts by the other two straight lines: shew that the locus of the point is the circumference of the circle circumscribing the square.

260. Circles are inscribed in the two triangles formed by drawing a perpendicular from an angle of a triangle on the opposite side; and analogous circles are described in relation to the two other like perpendiculars: shew that the sum of the diameters of the six circles together with the sum of the sides of the original triangle is equal to twice the sum of the three perpendiculars.

261. Three concentric circles are drawn in the same plane: draw a straight line, such that one of its segments between the inner and outer circumference may be bisected at one of the points at which the straight line meets the middle circumference.

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